TURBO
Continuous change

سلايدات :

تصميم تجارب هندسية

Design Of Experiment

للدكتور محـمـد الـجـراح

اللجنة الأكاديمية لقسم الهندسة الصناعية

2025

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Design and Analysis of Experiments

Textbooks:

- *Applied Statistics and Probability for Engineers*, D.C. Montgomery and G.C. Runger.
- Design and Analysis of Experiments, Montgomery D.C. (2009), 7th Ed., ISBN: 978-0470-39882-1, John Wiley and Sons, N.Y

Lecture: Monday & Wednesday 11:30-13:00

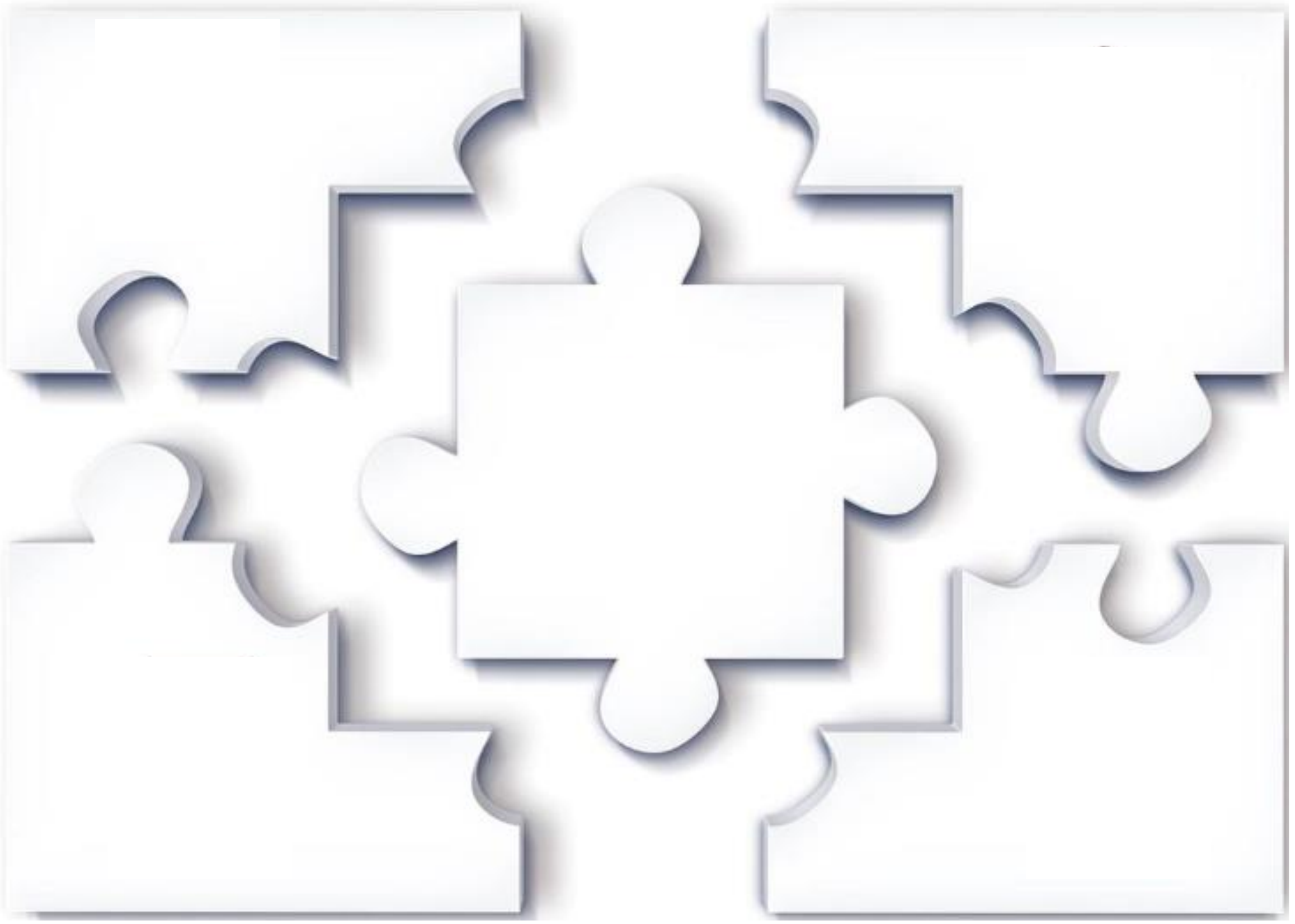
Instructor: *Prof. Mohammad Aljarrah, Ph.D., P. Eng., CLSSBB*

Midterm	25%
Project	25%
Class activities	10%
Final	40%

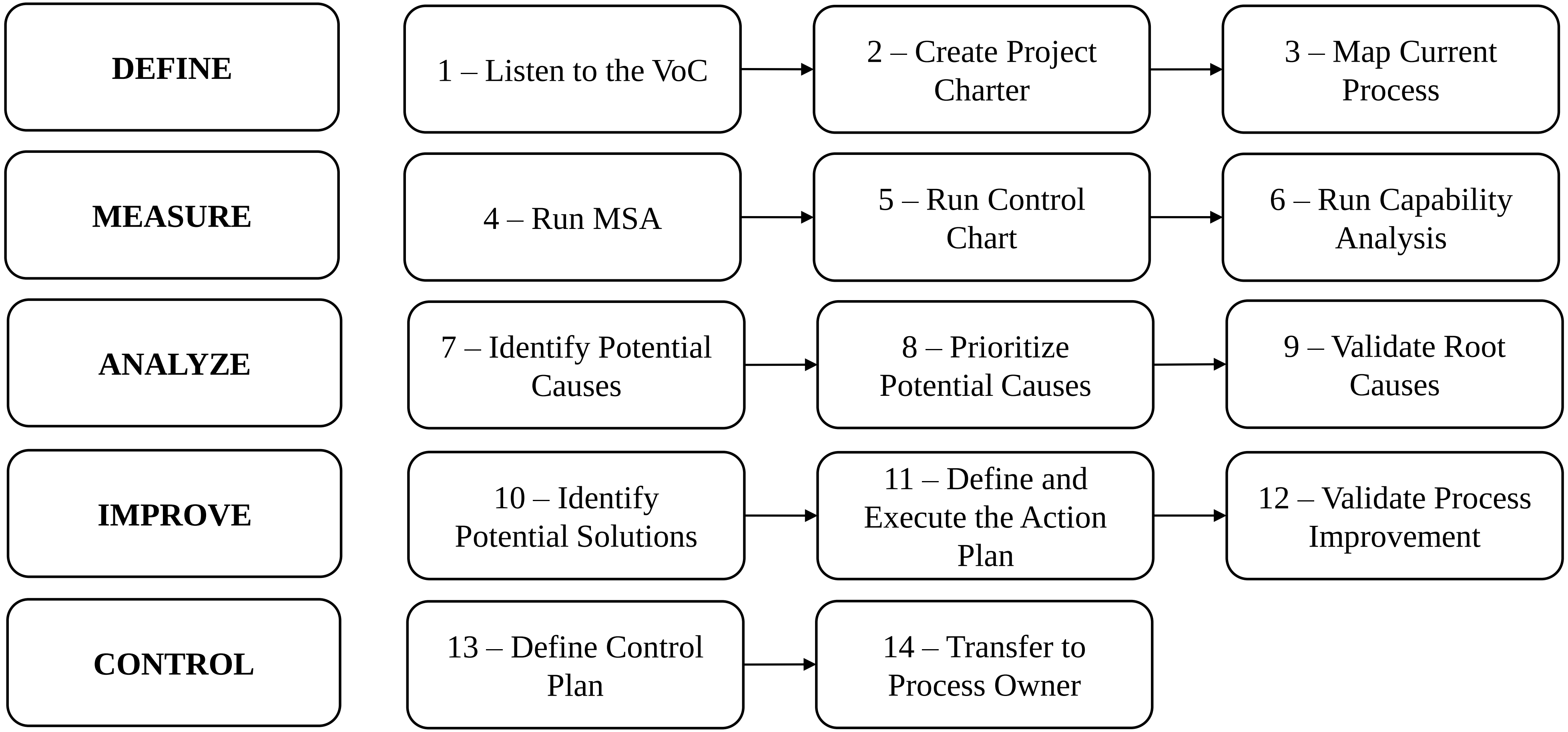
Course Outlines

- Review
- Linear and Multiple linear regressions
- Fixed-Effect Model
- Random-Effects Model
- Randomized Complete Block Model
- Two-Factor Factorial Experiments
- General Factorial Experiments
- 2^k Factorial Designs
- Blocking and Confounding in the 2^k Design
- Fractional Replication of the 2^k Design

Minitab®



DMAIC Roadmap



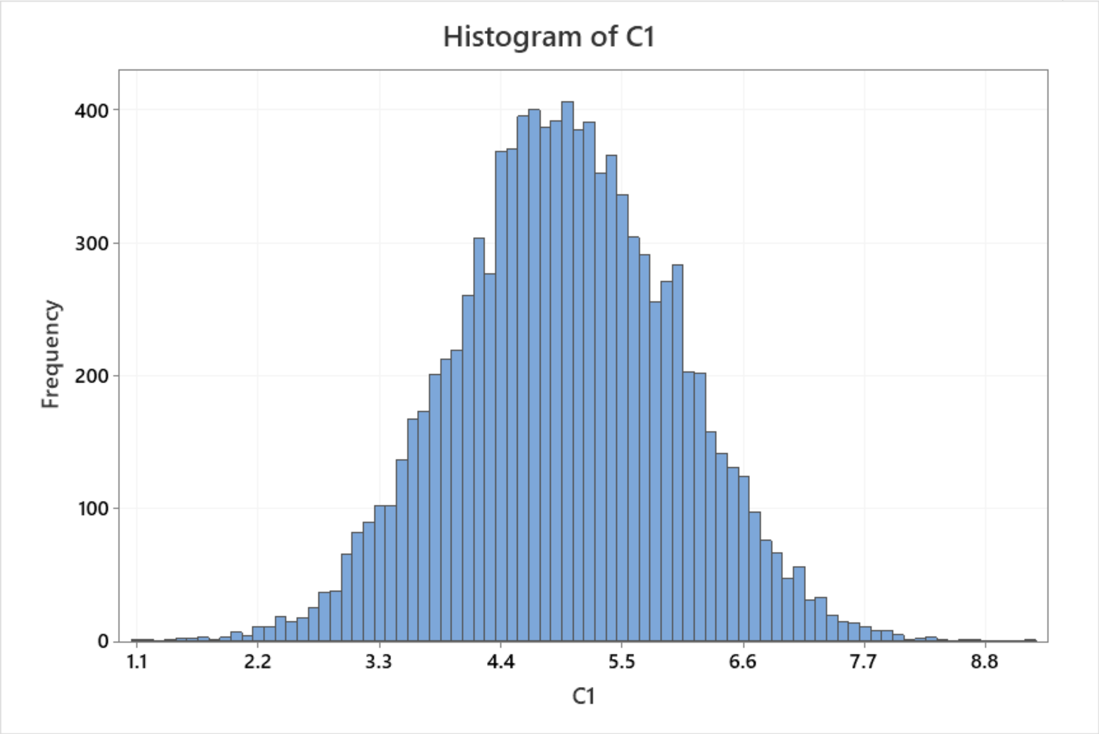
Review

- Sampling
- Normality
- Hypothesis testing
- *P*-Value
- Process Stability
- Measurement system analysis

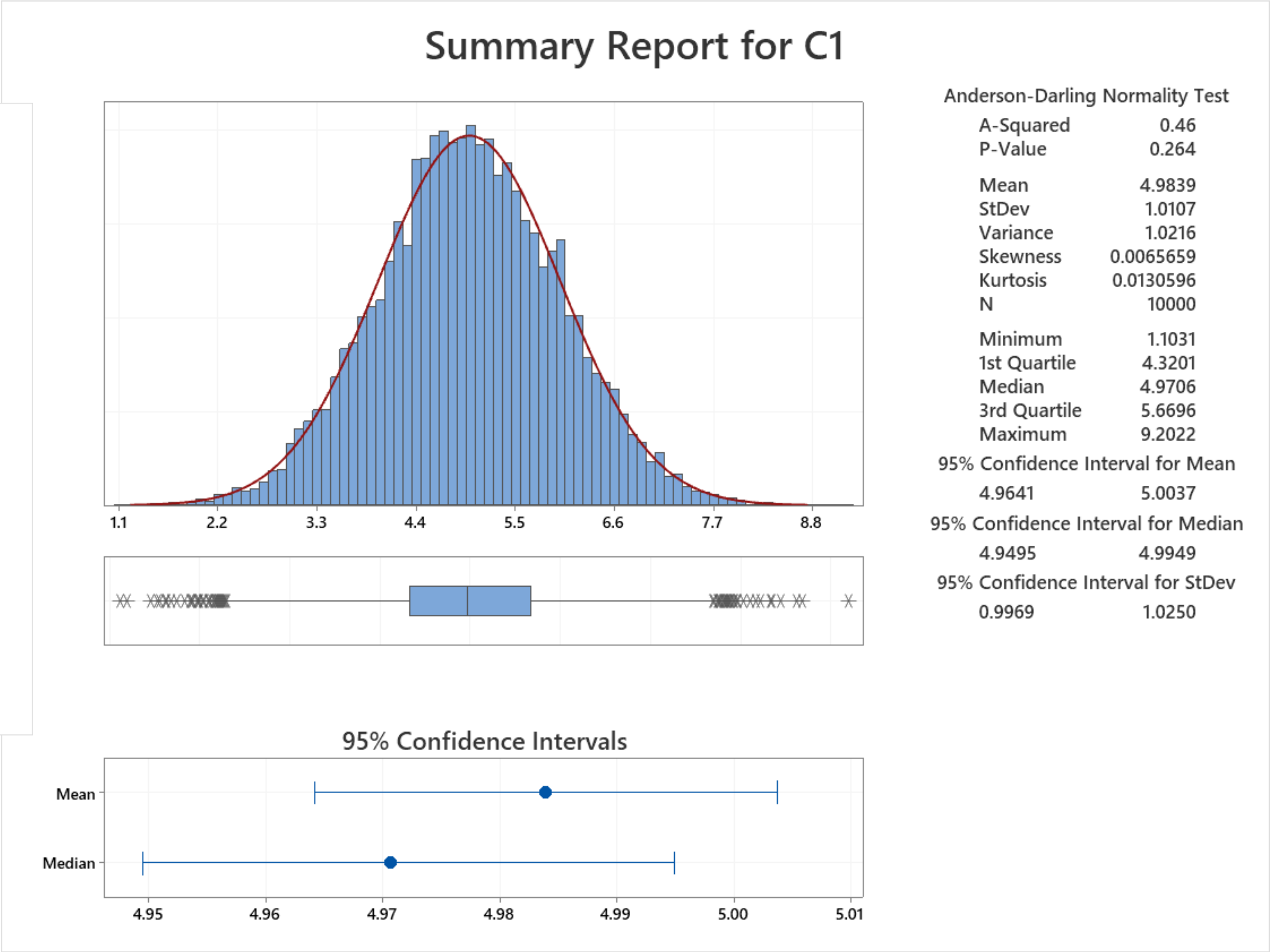
Sampling

- How to collect sample data?
- What is the sample size?
- What are types of samples?

Normality



Histogram of C1
Frequency
400
300
200
100
0
1.1
2.2
3.3
4.4
5.5
6.6
7.7
8.8
C1



Summary Report for C1
Anderson-Darling Normality Test
A-Squared 0.46
P-Value 0.264
Mean 4.9839
StDev 1.0107
Variance 1.0216
Skewness 0.0065659
Kurtosis 0.0130596
N 10000
Minimum 1.1031
1st Quartile 4.3201
Median 4.9706
3rd Quartile 5.6696
Maximum 9.2022
95% Confidence Interval for Mean
4.9641 5.0037
95% Confidence Interval for Median
4.9495 4.9949
95% Confidence Interval for StDev
0.9969 1.0250
1.1
2.2
3.3
4.4
5.5
6.6
7.7
8.8
95% Confidence Intervals
Mean
Median
4.95
4.96
4.97
4.98
4.99
5.00
5.01

Hypothesis testing

Descriptive Statistics

N	Mean	SE Mean	95% CI for μ
100	10.500	0.200	(10.108, 10.892)

μ: population mean of Sample
Known standard deviation = 2

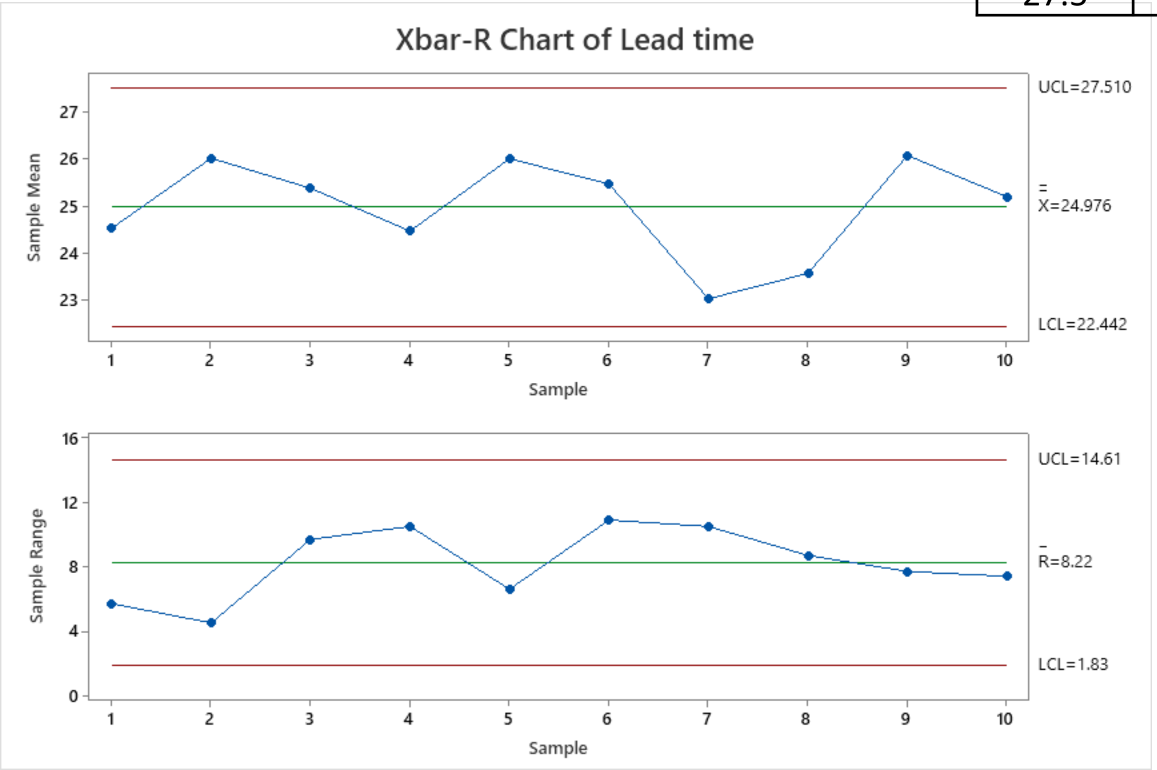
Test

Null hypothesis	H_0: $\mu = 10$
Alternative hypothesis	H_1: $\mu \neq 10$

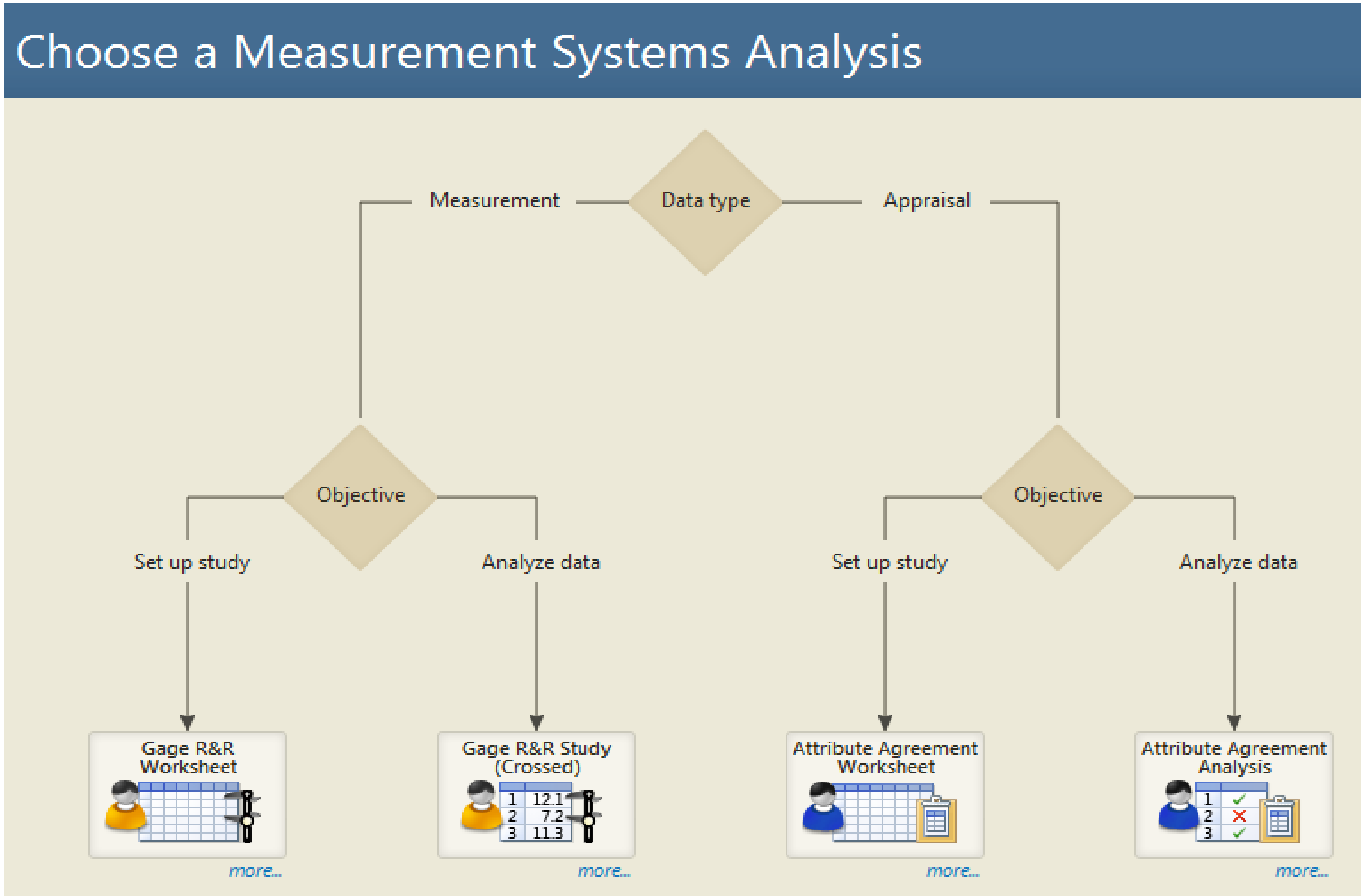
Z-Value	P-Value
2.50	0.012

Stability-Control Chart

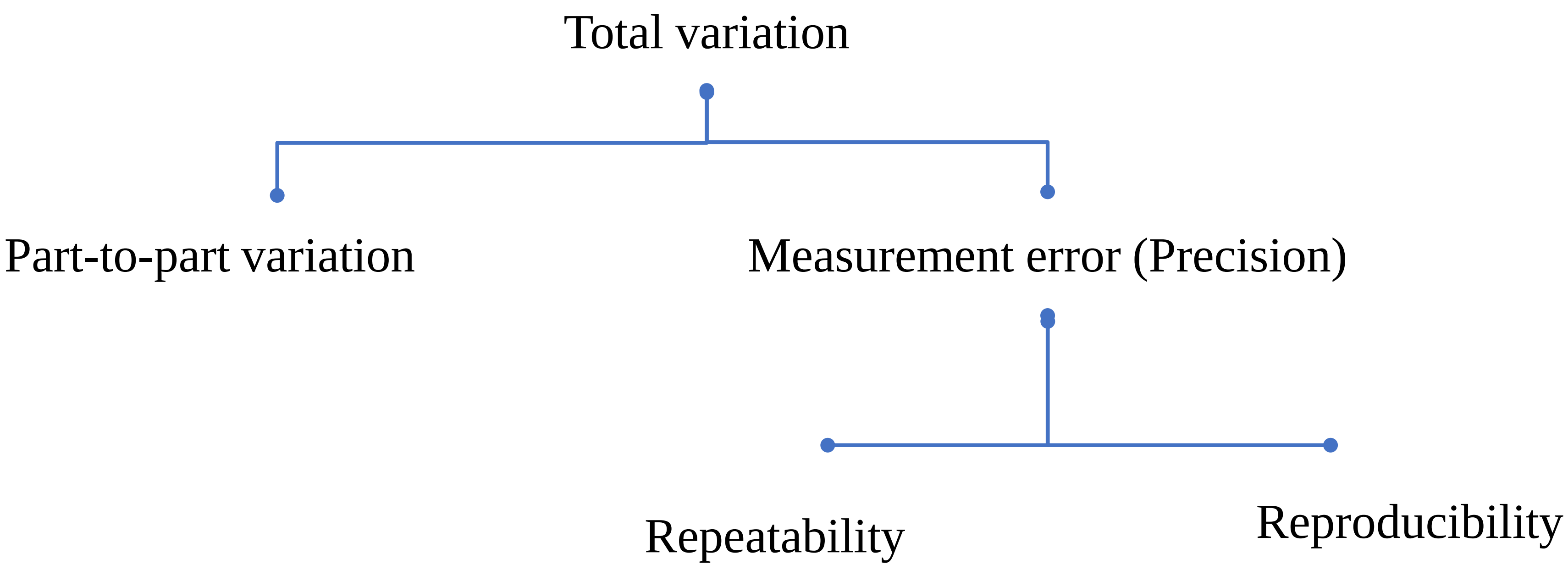
Day 1	Day 2	Day 3	Day 4	Day 5	Day 6	Day 7	Day 8	Day 9	Day 10
28.0	28.7	27.9	25.5	22.2	29.1	25.8	21.9	26.3	26.0
26.8	24.2	27.6	16.7	22.9	24.4	19.5	20.4	28.1	24.2
24.7	27.2	22.2	24.4	27.9	32.2	25.8	26.1	26.4	23.8
22.4	24.3	27.7	21.1	26.3	21.3	23.1	23.7	26.9	29.7
23.1	25.9	24.9	27.2	27.7	25.3	21.0	29.1	28.0	23.9
22.3	27.4	31.6	26.2	25.1	24.0	21.4	23.9	26.3	22.3
24.1	24.2	21.9	27.1	23.0	23.2	23.9	25.3	28.8	25.0
22.5	28.4	22.2	25.0	27.6	25.1	18.6	23.3	24.9	25.3
23.9	25.4	24.5	27.2	28.6	24.9	29.1	21.1	21.1	25.9
27.5	24.5	23.3	24.3	28.8	25.2	22.1	20.9	24.0	25.9

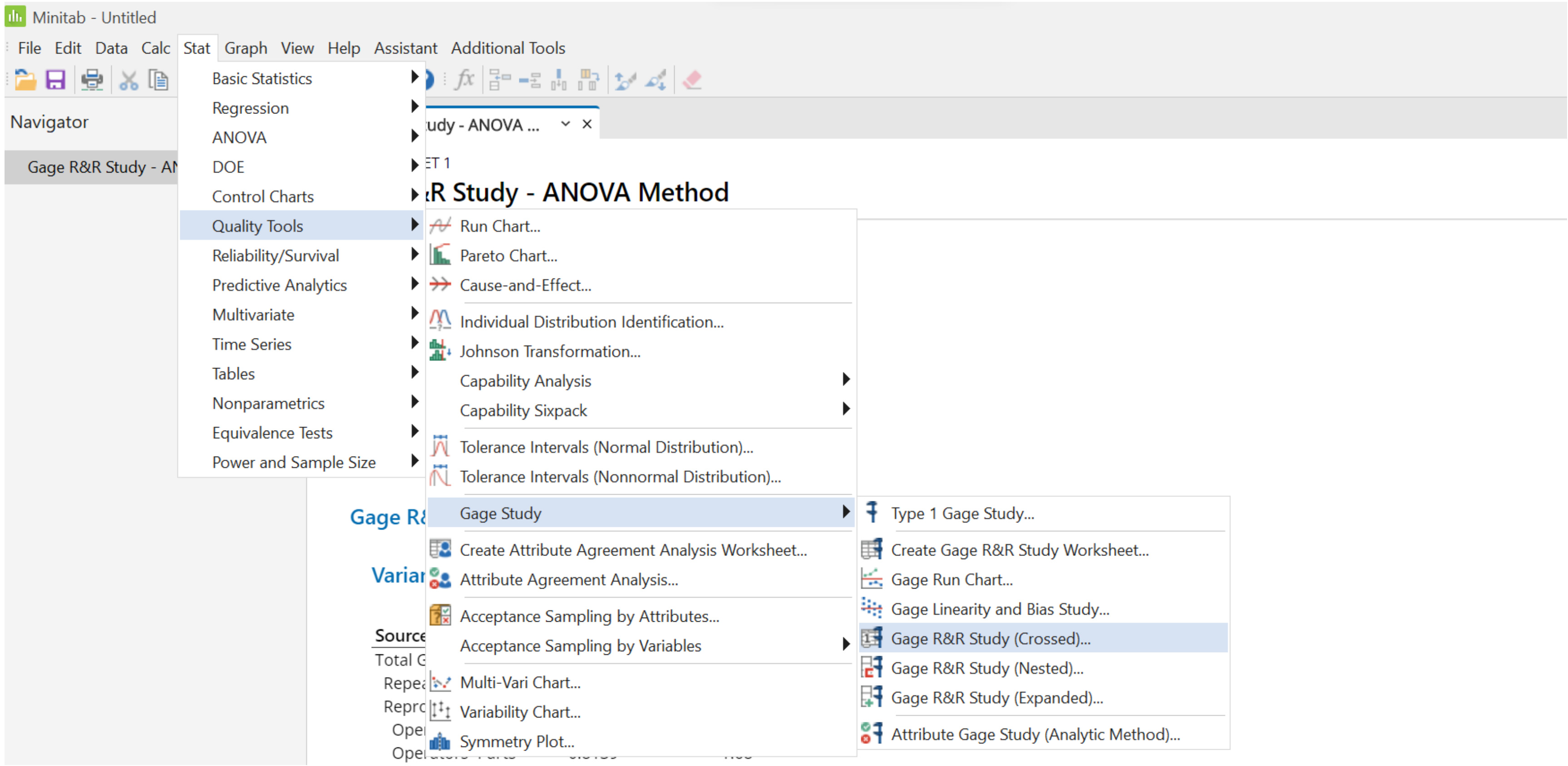


Measurement system analysis



Repeatability & Reproducibility (R&R)




Minitab - Untitled
File Edit Data Calc Stat Graph View Help Assistant Additional Tools
Navigator
Basic Statistics
Regression
ANOVA
DOE
Control Charts
Quality Tools
Reliability/Survival
Predictive Analytics
Multivariate
Time Series
Tables
Nonparametrics
Equivalence Tests
Power and Sample Size
R Study - ANOVA Method
Run Chart...
Pareto Chart...
Cause-and-Effect...
Individual Distribution Identification...
Johnson Transformation...
Capability Analysis
Capability Sixpack
Tolerance Intervals (Normal Distribution)...
Tolerance Intervals (Nonnormal Distribution)...
Gage Study
Create Attribute Agreement Analysis Worksheet...
Attribute Agreement Analysis...
Acceptance Sampling by Attributes...
Acceptance Sampling by Variables
Multi-Vari Chart...
Variability Chart...
Symmetry Plot...
Type 1 Gage Study...
Create Gage R&R Study Worksheet...
Gage Run Chart...
Gage Linearity and Bias Study...
Gage R&R Study (Crossed)...
Gage R&R Study (Nested)...
Gage R&R Study (Expanded)...
Attribute Gage Study (Analytic Method)...

Parts	Operators	Measurement data
Part 1	Operator 1	13
Part 1	Operator 1	13
Part 2	Operator 1	16
Part 2	Operator 1	15
Part 3	Operator 1	15
Part 3	Operator 1	16
Part 4	Operator 1	15
Part 4	Operator 1	15
Part 5	Operator 1	20
Part 5	Operator 1	20
Part 6	Operator 1	22
Part 6	Operator 1	22
Part 7	Operator 1	24
Part 7	Operator 1	25
Part 8	Operator 1	27
Part 8	Operator 1	27
Part 9	Operator 1	29
Part 9	Operator 1	29
Part 10	Operator 1	36
Part 10	Operator 1	35
Part 1	Operator 2	10
Part 1	Operator 2	10
Part 2	Operator 2	12
Part 2	Operator 2	13
Part 3	Operator 2	13
Part 3	Operator 2	12
Part 4	Operator 2	15
Part 4	Operator 2	15
Part 5	Operator 2	16
Part 5	Operator 2	18
Part 6	Operator 2	20
Part 6	Operator 2	20
Part 7	Operator 2	22
Part 7	Operator 2	21
Part 8	Operator 2	21
Part 8	Operator 2	22
Part 9	Operator 2	27
Part 9	Operator 2	26
Part 10	Operator 2	28
Part 10	Operator 2	30
Part 1	Operator 3	11
Part 1	Operator 3	11
Part 2	Operator 3	13
Part 2	Operator 3	13
Part 3	Operator 3	12
Part 3	Operator 3	15
Part 4	Operator 3	12
Part 4	Operator 3	12
Part 5	Operator 3	17
Part 5	Operator 3	18
Part 6	Operator 3	19
Part 6	Operator 3	20
Part 7	Operator 3	21
Part 7	Operator 3	22
Part 8	Operator 3	23
Part 8	Operator 3	24
Part 9	Operator 3	28
Part 9	Operator 3	28
Part 10	Operator 3	31
Part 10	Operator 3	27

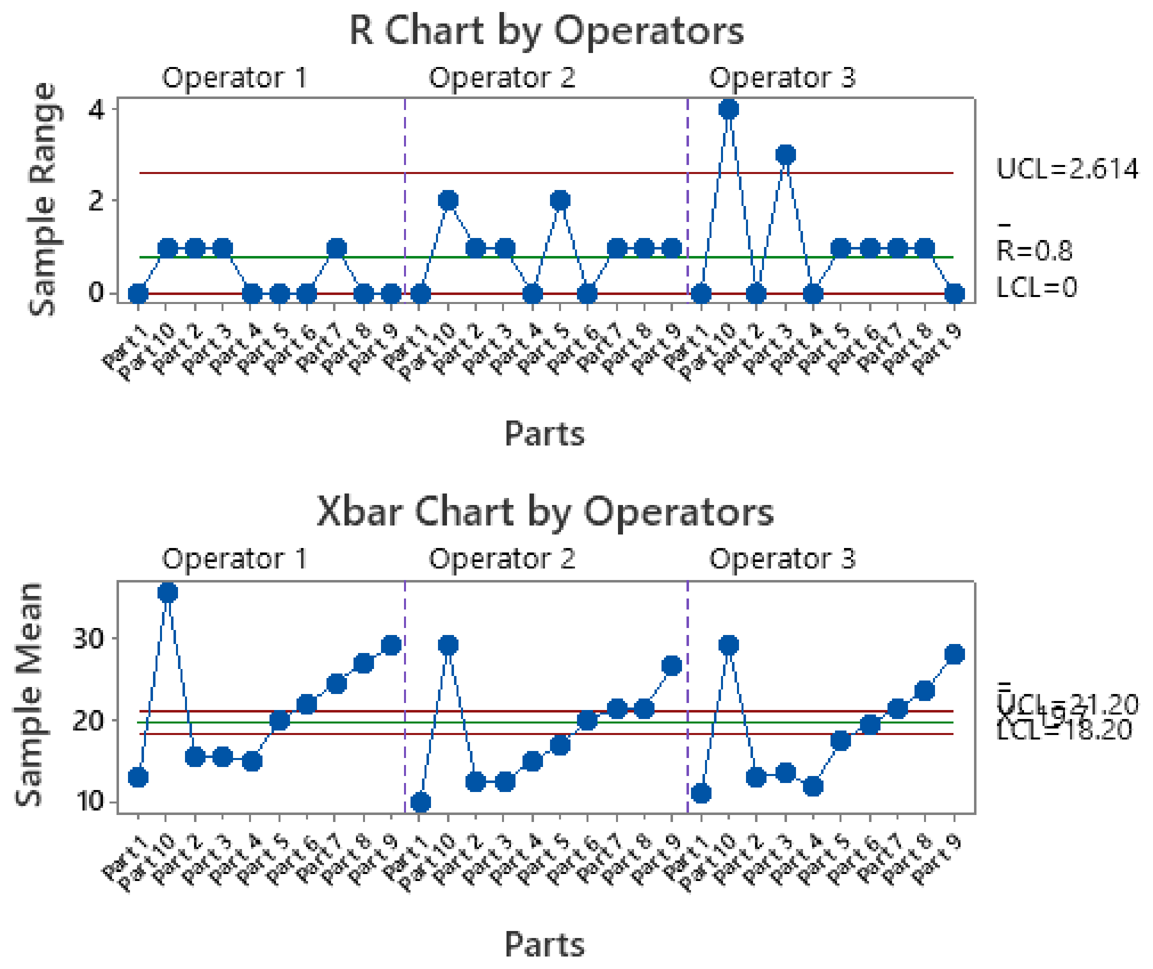
Gage Evaluation

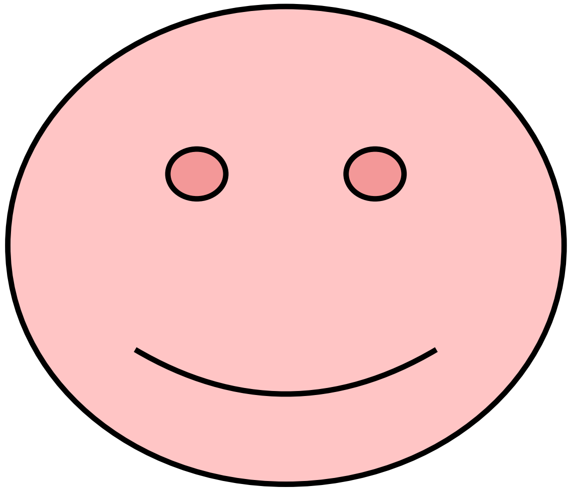
Source	StdDev (SD)	Study Var (6 × SD)	%Study Var (%SV)
Total Gage R&R	2.11739	12.7043	30.39
Repeatability	0.87560	5.2536	12.57
Reproducibility	1.92787	11.5672	27.67
Operators	1.70375	10.2225	24.45
Operators*Parts	0.90216	5.4129	12.95
Part-To-Part	6.63890	39.8334	95.27
Total Variation	6.96838	41.8103	100.00

- ❑ Total Gage R&R should be less than 10%
- ❑ Between 10 and 30% is marginable, there is room for improvement
- ❑ More than 30%, measurement system is not acceptable

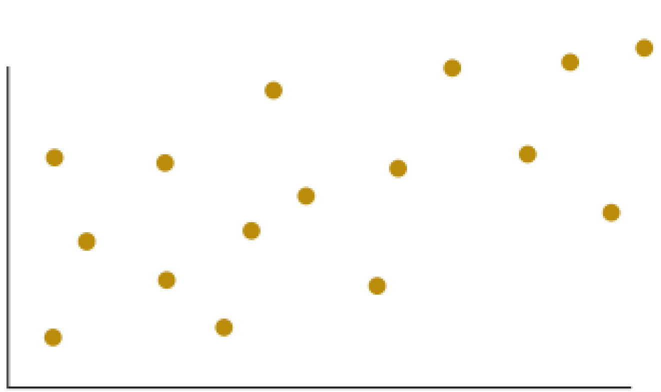
R-Chart=Repeatability. We wish to have points on zero line.

Xbar-Chart=Reproducibility. We wish to have shape as similar as possible. It is ok to have all points outside the control limit.

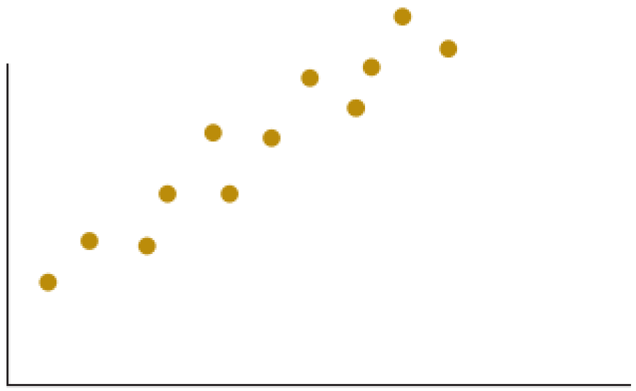




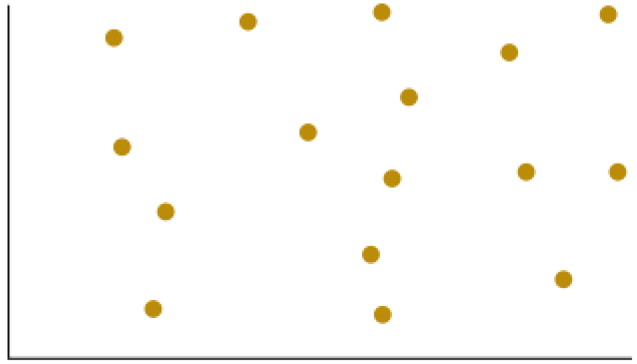
Simple Linear Regression and Correlation



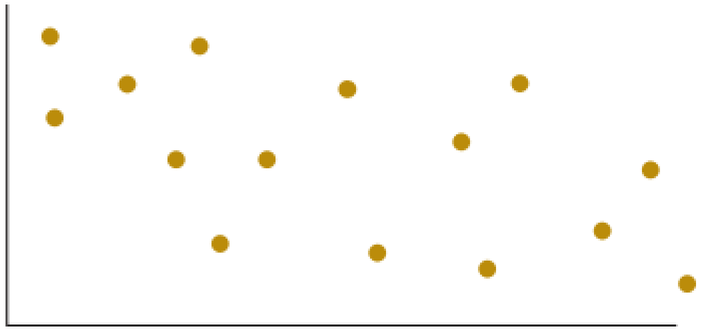
(*a*) Weak positive relationship



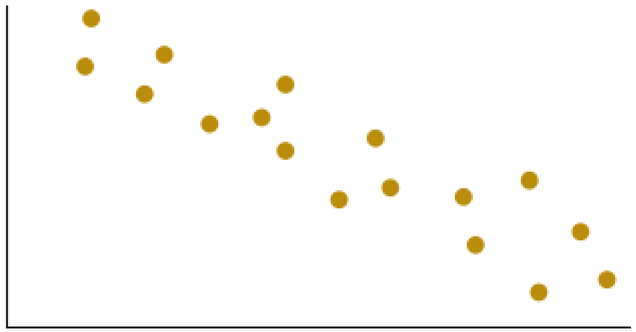
(*b*) Strong positive relationship



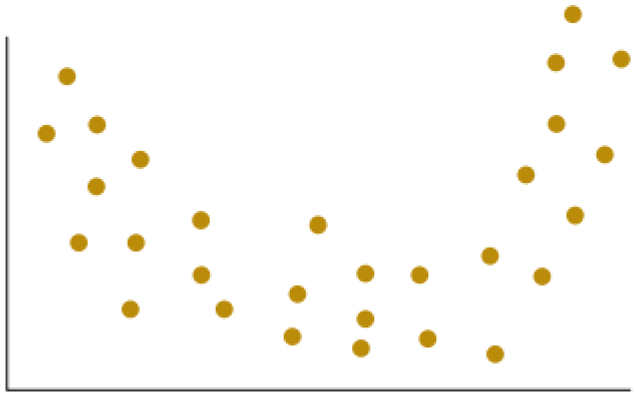
(*f*) No relationship, $r_{xy} \approx 0$



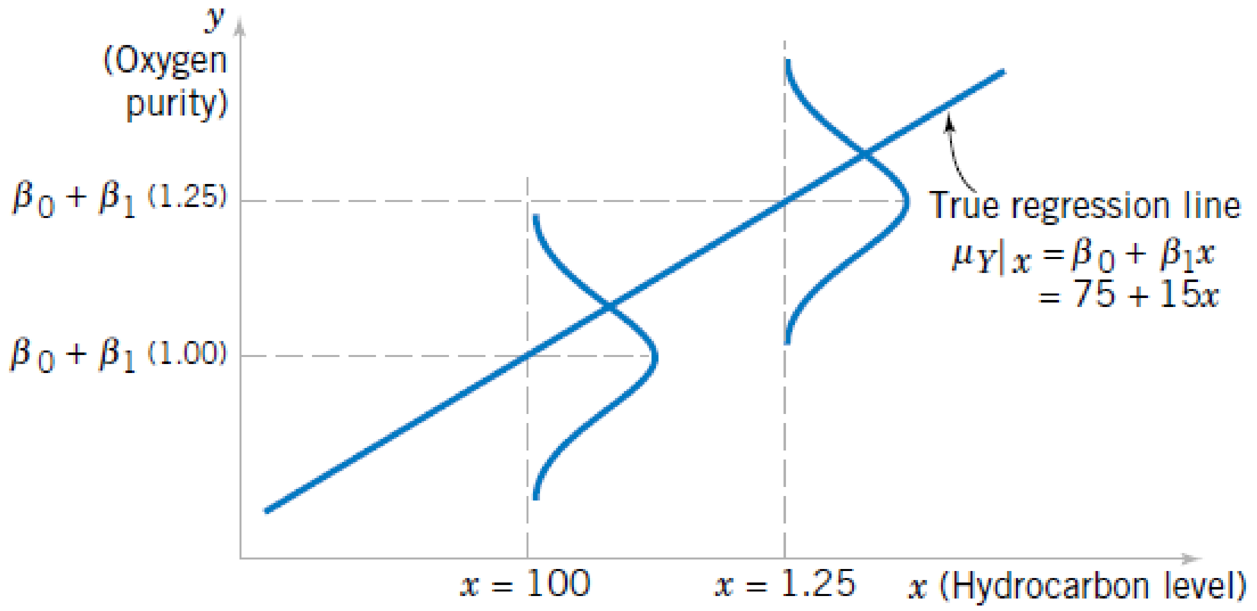
(*c*) Weak negative relationship



(*d*) Strong negative relationship



(*e*) Nonlinear quadratic relationship, $r_{xy} \approx 0$



y
(Oxygen
purity)
$\beta_0 + \beta_1$ (1.25)
$\beta_0 + \beta_1$ (1.00)
True regression line
$\mu_{Y|x} = \beta_0 + \beta_1 x$
$= 75 + 15x$
$x = 100$
$x = 1.25$
x (Hydrocarbon level)

- ✓ The case of **simple linear regression** considers a single **regressor** or **predictor** "x" and a dependent or response Variable "Y".

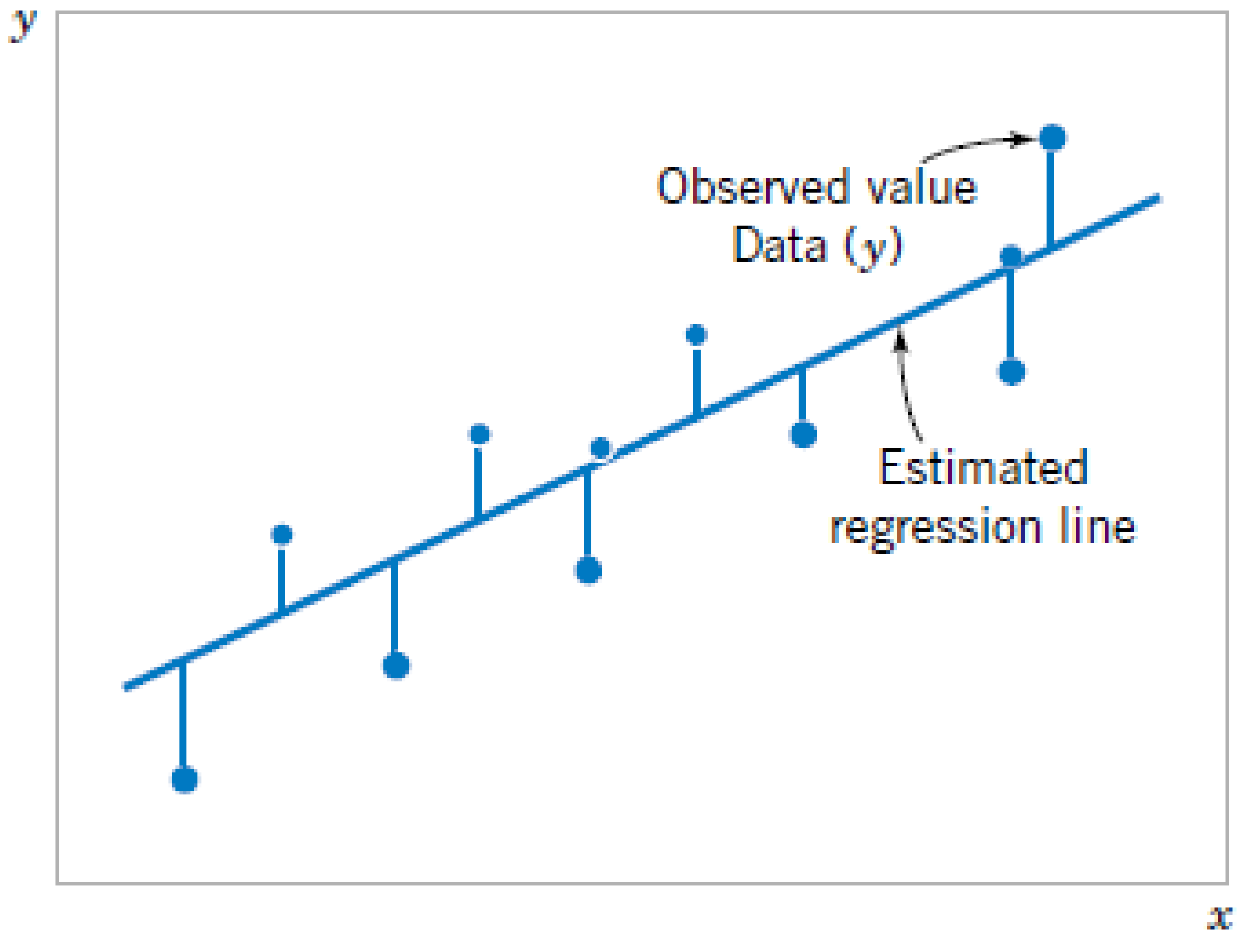
- ✓ Suppose that the true relationship between "Y" and "x" is a straight line and that the observation "Y" at each level of "x" is a random variable

$$\mathrm{E}(Y|x) = \beta_0 + \beta_1 x$$

- ✓ Assume each observation is represented by a model

$$y(x) = \beta_0 + \beta_1 x + \epsilon$$

$$Where\ \epsilon\ is\ a\ random\ error\ with\ mean\ zero\ and\ variance\ \sigma^2$$



The sum of squares of the deviations of the observations from the true regression line is

$$L = \sum_{i=1}^{n} {\epsilon_i}^{2} = \sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x)^2$$

$$\left.\frac{\partial L}{\partial \beta_0}\right|_{\hat{\beta}_0,\hat{\beta}_1} = -2\sum_{i=1}^{n}(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i) = 0$$

$$L = \sum_{i=1}^{n} \epsilon_i^{\ 2} = \sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x)^2$$

$$\left.\frac{\partial L}{\partial \beta_1}\right|_{\hat{\beta}_0,\hat{\beta}_1} = -2\sum_{i=1}^{n}(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)x_i = 0$$

$$n\hat{\beta}_0 + \hat{\beta}_1 \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$$

$$\hat{\beta}_0 \sum_{i=1}^{n} x_i + \hat{\beta}_1 \sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} y_i x_i$$

Definition

The **least squares estimates** of the intercept and slope in the simple linear regression model are

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x} \tag{11-7}$$

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n} y_i x_i - \dfrac{\left(\sum_{i=1}^{n} y_i\right)\left(\sum_{i=1}^{n} x_i\right)}{n}}{\sum_{i=1}^{n} x_i^2 - \dfrac{\left(\sum_{i=1}^{n} x_i\right)^2}{n}} \tag{11-8}$$

where $\bar{y} = (1/n)\sum_{i=1}^{n} y_i$ and $\bar{x} = (1/n)\sum_{i=1}^{n} x_i$.

The **fitted** or **estimated regression line** is therefore

$$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x$$

$$y_i = \hat{\beta}_0 + \hat{\beta}_1 x_i + e_i, \qquad i = 1, 2, \ldots, n$$

where $e_i = y_i - \hat{y}_i$ is called the **residual.**

- ✓ The residual describes the error in the fit of the model to the i^{th} observation of y_i

$$S_{xx} = \sum_{i=1}^{n} (x_i - \bar{x})^2 = \sum_{i=1}^{n} x_i^2 - \frac{\left(\sum_{i=1}^{n} x_i\right)^2}{n}$$

$$S_{xy} = \sum_{i=1}^{n} y_i (x_i - \bar{x})^2 = \sum_{i=1}^{n} x_i y_i - \frac{\left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right)}{n}$$

The following data represents *y* is the purity of oxygen produced in a chemical distillation process, and *x* is the percentage of hydrocarbons present in the main condenser of the distillation unit. Construct the relationship between oxygen purity and percentage of hydrogen and optimize the process.

Observation Number	Hydrocarbon Level x (%)	Purity y (%)
1	0.99	90.01
2	1.02	89.05
3	1.15	91.43
4	1.29	93.74
5	1.46	96.73
6	1.36	94.45
7	0.87	87.59
8	1.23	91.77
9	1.55	99.42
10	1.40	93.65
11	1.19	93.54
12	1.15	92.52
13	0.98	90.56
14	1.01	89.54
15	1.11	89.85
16	1.20	90.39
17	1.26	93.25
18	1.32	93.41
19	1.43	94.98
20	0.95	87.33

$$n = 20 \quad \sum_{i=1}^{20} x_i = 23.92 \quad \sum_{i=1}^{20} y_i = 1{,}843.21 \quad \overline{x} = 1.1960 \quad \overline{y} = 92.1605$$

$$\sum_{i=1}^{20} y_i^2 = 170{,}044.5321 \quad \sum_{i=1}^{20} x_i^2 = 29.2892 \quad \sum_{i=1}^{20} x_i y_i = 2{,}214.6566$$

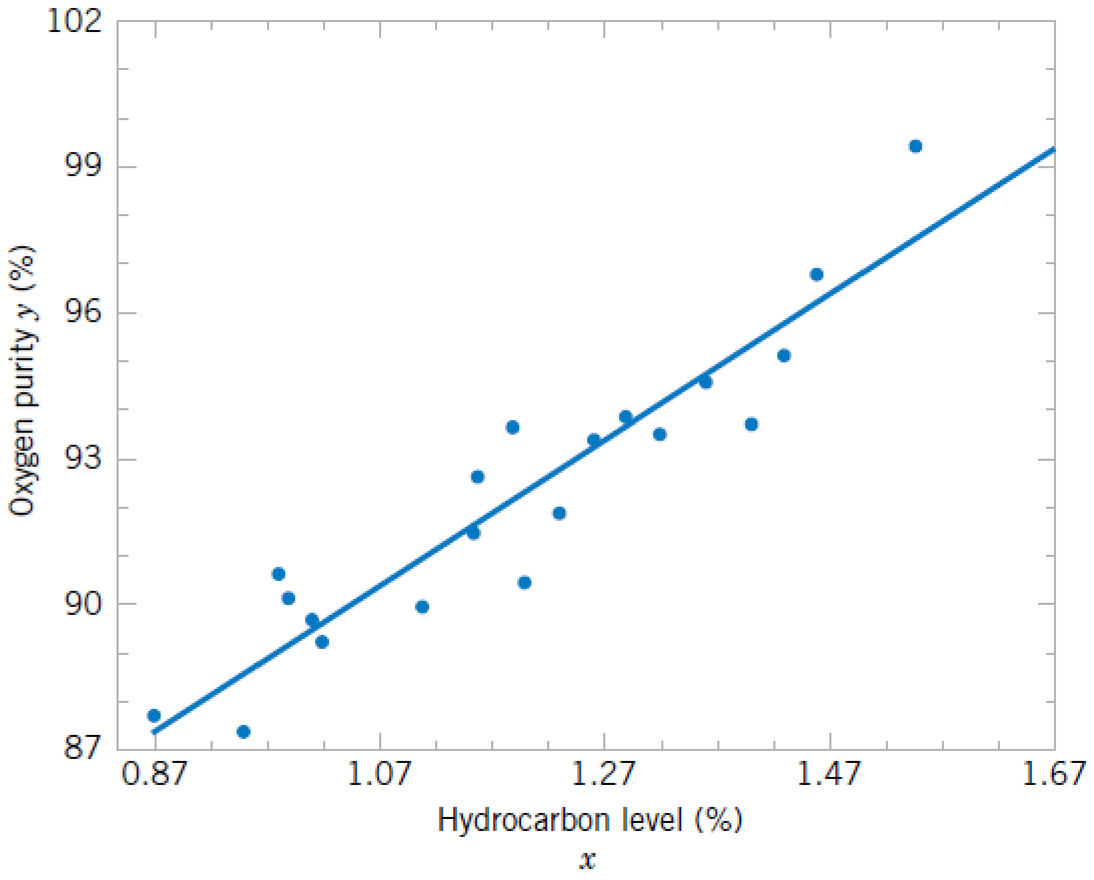
$$S_{xx} = \sum_{i=1}^{20} x_i^2 - \frac{\left(\sum_{i=1}^{20} x_i\right)^2}{20} = 29.2892 - \frac{(23.92)^2}{20} = 0.68088$$

$$S_{xy} = \sum_{i=1}^{20} x_i y_i - \frac{\left(\sum_{i=1}^{20} x_i\right)\left(\sum_{i=1}^{20} y_i\right)}{20} = 2{,}214.6566 - \frac{(23.92)(1{,}843.21)}{20} = 10.17744$$

$$\hat{\beta}_1 = \frac{S_{xy}}{S_{xx}} = \frac{10.17744}{0.68088} = 14.94748$$

$$\hat{\beta}_0 = \overline{y} - \hat{\beta}_1 \overline{x} = 92.1605 - (14.94748)1.196 = 74.28331$$

$$\hat{y} = 74.283 + 14.947x$$


102
99
96
93
90
87
0.87
1.07
1.27
1.47
1.67
Oxygen purity y (%)
Hydrocarbon level (%)
x

Estimator of σ^2

- ✓ There is another unknown parameter in the regression model (σ^2) “the variance of the error term ε.
- ✓ The residual $e_i = y_i - \hat{y}_i$ are used to obtain an estimate of σ^2
- ✓ The sum squares of the residuals is called **error sum of squares**

$$SS_E = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} (y_i - \hat{y}_i)^2$$

Therefore an **unbiased estimator** of σ^2 is

$$\hat{\sigma}^2 = \frac{SS_E}{n - 2}$$

$$SS_E = SS_T - \hat{\beta}_1 S_{xy}$$

Is the total sum of squares of the response variable *y*

where $SS_T = \sum_{i=1}^{n} (\hat{y}_i - \bar{y})^2 = \sum_{i=1}^{n} y_i^2 - n\bar{y}^2$

Example

Y	240	181	193	155	172	110	113	75	94
X	1.6	9.4	15.5	20	22	35.5	43	40.5	33

a) Fit the simple linear regression model using least squares method
b) Find an estimator of σ^2
c) Predict wear when viscosity x = 30
d) Obtain the fitted value of y when x = 22 and calculate the corresponding residual

$$\sum x_i = 220.5, \qquad \sum x_i^{\;2} = 7053.67 \qquad \sum y_i = 1333$$

$$\sum y_i x_i = 26864.4$$

a) *y=234.1 - 3.509 x*

b) σ^2 *=398.3*

b) *y at x =30, y=128.83*

b) *y at x =22, y=156.902*
Residual=172-156.902=-15.098

a) $y=234.1 - 3.509\,x$

Properties of the least square estimators

$$E(\hat{\beta}_1) = \beta_1$$

Thus, $\hat{\beta}_1$ is an **unbiased estimator** of the true slope β_1

$$V(\hat{\beta}_1) = \frac{\sigma^2}{S_{xx}}$$

$$E(\hat{\beta}_0) = \beta_0 \quad \text{and} \quad V(\hat{\beta}_0) = \sigma^2 \left[\frac{1}{n} + \frac{\bar{x}^2}{S_{xx}} \right]$$

Definition

In simple linear regression the **estimated standard error of the slope** and the **estimated standard error of the intercept** are

$$se(\hat{\beta}_1) = \sqrt{\frac{\hat{\sigma}^2}{S_{xx}}} \quad \text{and} \quad se(\hat{\beta}_0) = \sqrt{\hat{\sigma}^2 \left[\frac{1}{n} + \frac{\bar{x}^2}{S_{xx}} \right]}$$

respectively, where $\hat{\sigma}^2$ is computed from Equation 11-13.

Hypothesis test in simple linear regression

Suppose we wish to test the hypothesis that the slope equals a constant, say, $\beta_{1,0}$

$$H_0: \beta_1 = \beta_{1,0}$$
$$H_1: \beta_1 \neq \beta_{1,0}$$

$$T_0 = \frac{\hat{\beta}_1 - \beta_{1,0}}{\sqrt{\hat{\sigma}^2/S_{xx}}}$$

We would reject the null hypothesis if $|t_0| > t_{\alpha/2,n-2}$

A similar procedure can be used to test hypotheses about the intercept. To test

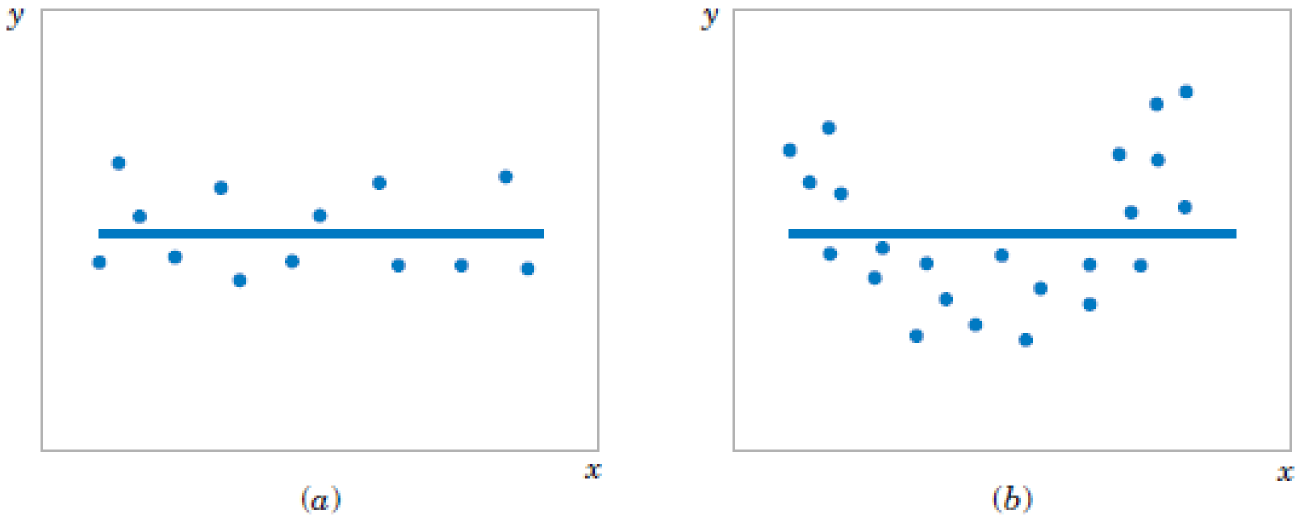
$$H_0: \beta_0 = \beta_{0,0}$$
$$H_1: \beta_0 \neq \beta_{0,0}$$

$$T_0 = \frac{\hat{\beta}_0 - \beta_{0,0}}{\sqrt{\hat{\sigma}^2\left[\frac{1}{n} + \frac{\bar{x}^2}{S_{xx}}\right]}} = \frac{\hat{\beta}_0 - \beta_{0,0}}{se(\hat{\beta}_0)}$$

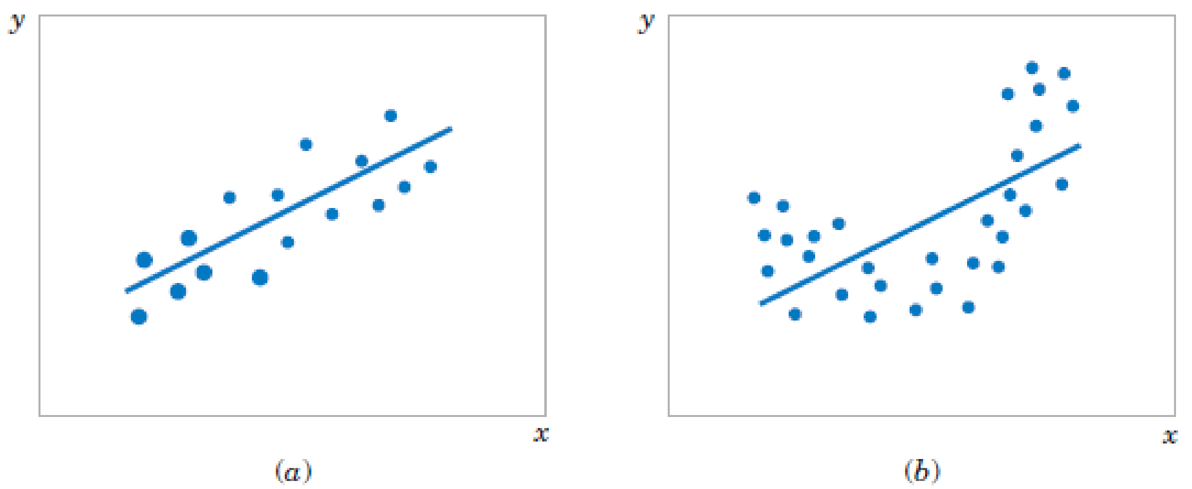
Special Case

$$H_0: \beta_1 = 0$$
$$H_1: \beta_1 \neq 0$$

Accept the null hypothesis is equivalent to conclude that there is no linear relationship between x and y.



Accept the null hypothesis



Reject the null hypothesis

$$H_0\colon \beta_1 = 0$$
$$H_1\colon \beta_1 \neq 0$$

Teams	Games Won (y)	Yards Rushing by Opponent (x)	Teams	Games Won (y)	Yards Rushing by Opponent (x)
Washington	10	2205	Detroit	6	1901
Minnesota	11	2096	Green Bay	5	2288
New England	11	1847	Houston	5	2072
Oakland	13	1903	Kansas City	5	2861
Pittsburgh	10	1457	Miami	6	2411
Baltimore	11	1848	New Orleans	4	2289
Los Angeles	10	1564	New York Giants	3	2203
Dallas	11	1821	New York Jets	3	2592
Atlanta	4	2577	Philadelphia	4	2053
Buffalo	2	2476	St. Louis	10	1979
Chicago	7	1984	San Diego	6	2048
Cincinnati	10	1917	San Francisco	8	1786
Cleveland	9	1761	Seattle	2	2876
Denver	9	1709	Tampa Bay	0	2560

a) Estimate the standard errors of the slope and intercept

b) Test (using $\alpha = 0.01$*)* H_0: $\beta_1 = -0.01$, H_1: $\beta_1 \neq -0.01$

$$\hat{y} = 21.7883 - 0.0070251$$

$$Se(\beta_1) = \sqrt{\frac{\hat{\sigma}^2}{S_{xx}}} = \sqrt{\frac{5.7257}{3608611.43}} = 0.001259$$

$$\mathrm{Se}(\beta_1) = \sqrt{\hat{\sigma}^2(\frac{1}{n} + \frac{\bar{x}^2}{S_{xx}})} = \sqrt{5.7257(\frac{1}{28} + \frac{2110.132}{3608611.43}} = 2.6962$$

$$t_0 = \frac{\hat{\beta}_1 \_ \beta_{1,0}}{Se(\hat{\beta}_1)} = \frac{-0.0070251 + 0.01}{0.001259} = 2.3618$$

$$t_{-\frac{\alpha}{2},n-2} = t_{-0.005,26} = 2.779$$

$$t_0 < t_{-\frac{\alpha}{2},n-2} \quad Accept\ H_0$$

Analysis of Variance Approach to Test Significance of Regression

- ✓ A method called the analysis of variance can be used to test the significance of regression
- ✓ The analysis of variance identity can be written as follow:

$$\sum_{i=1}^{n}(y_i - \bar{y})^2 = \sum_{i=1}^{n}(\hat{y}_i - \bar{y})^2 + \sum_{i=1}^{n}(y_i - \hat{y}_i)^2$$

SS_R: Regression sum of squares

SS_E: Error sum of squares

Total corrected sum of squares

$SS_T = SS_R + SS_E$

Analysis of Variance Approach to Test Significance of Regression

Table 11-3 Analysis of Variance for Testing Significance of Regression

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0
Regression	$SS_R = \hat{\beta}_1 S_{xy}$	1	MS_R	MS_R/MS_E
Error	$SS_E = SS_T - \hat{\beta}_1 S_{xy}$	$n - 2$	MS_E	
Total	SS_T	$n - 1$		

Note that $MS_E = \hat{\sigma}^2$.

$$F_0 = \frac{SS_R/1}{SS_E/(n-2)} = \frac{MS_R}{MS_E}$$

Table 11-1 Oxygen and Hydrocarbon Levels

Observation Number	Hydrocarbon Level x(%)	Purity y(%)
1	0.99	90.01
2	1.02	89.05
3	1.15	91.43
4	1.29	93.74
5	1.46	96.73
6	1.36	94.45
7	0.87	87.59
8	1.23	91.77
9	1.55	99.42
10	1.40	93.65
11	1.19	93.54
12	1.15	92.52
13	0.98	90.56
14	1.01	89.54
15	1.11	89.85
16	1.20	90.39
17	1.26	93.25
18	1.32	93.41
19	1.43	94.98
20	0.95	87.33

$$SS_T = 173.38, \; \hat{\beta}_1 = 14.947,$$

$$S_{xy} = 10.17744$$

$$SS_R = \hat{\beta}_1 S_{xy} = (14.947)10.17744 = 152.13$$

$$SS_E = SS_T - SS_R = 173.38 - 152.13 = 21.25$$

The analysis of variance for testing $H_0\colon \beta_1 = 0$

The test statistic is $f_0 = MS_R/MS_E = 152.13/1.18 = 128.86$,

that the P-value is $P \simeq 1.23 \times 10^{-9}$, so we conclude that β_1 is not zero.

The regression equation is $Y = \beta_0 + \beta_1 x$				
Predictor	**Coef**	**Se Coef**	**T**	**P**
Constant	12.857	1.032	?	?
X	2.3445	0.115	?	

Analysis of Variance				
Source	**DF**	**SS**	**MS**	**F**
Regression	1	912.43	?	?
Residual error	8	17.55	?	
Total	9	929.98		

(a) Fill in the missing information. You may use bounds for the P-values.

(b) Can you conclude that the model defines a useful linear relationship?

(c) What is your estimate of σ^2

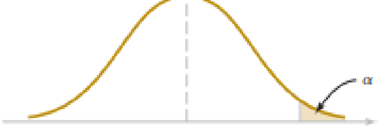
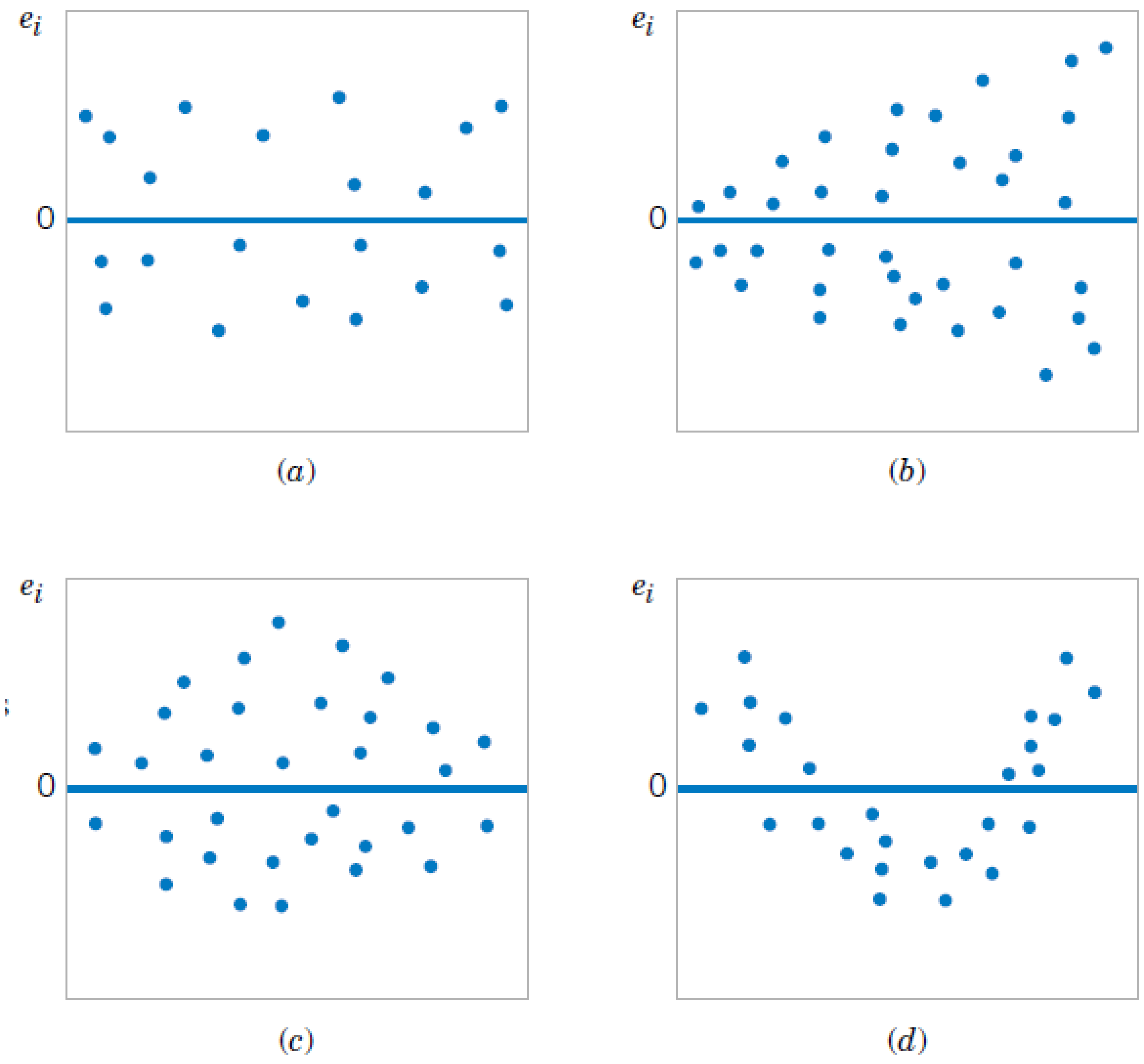


TABLE • V Percentage Points $t_{\alpha,\nu}$ of the t Distribution

ν \ α	.40	.25	.10	.05	.025	.01	.005	.0025	.001	.0005
1	.325	1.000	3.078	6.314	12.706	31.821	63.657	127.32	318.31	636.62
2	.289	.816	1.886	2.920	4.303	6.965	9.925	14.089	23.326	31.598
3	.277	.765	1.638	2.353	3.182	4.541	5.841	7.453	10.213	12.924
4	.271	.741	1.533	2.132	2.776	3.747	4.604	5.598	7.173	8.610
5	.267	.727	1.476	2.015	2.571	3.365	4.032	4.773	5.893	6.869
6	.265	.718	1.440	1.943	2.447	3.143	3.707	4.317	5.208	5.959
7	.263	.711	1.415	1.895	2.365	2.998	3.499	4.029	4.785	5.408
8	.262	.706	1.397	1.860	2.306	2.896	3.355	3.833	4.501	5.041
9	.261	.703	1.383	1.833	2.262	2.821	3.250	3.690	4.297	4.781
10	.260	.700	1.372	1.812	2.228	2.764	3.169	3.581	4.144	4.587
11	.260	.697	1.363	1.796	2.201	2.718	3.106	3.497	4.025	4.437
12	.259	.695	1.356	1.782	2.179	2.681	3.055	3.428	3.930	4.318
13	.259	.694	1.350	1.771	2.160	2.650	3.012	3.372	3.852	4.221
14	.258	.692	1.345	1.761	2.145	2.624	2.977	3.326	3.787	4.140
15	.258	.691	1.341	1.753	2.131	2.602	2.947	3.286	3.733	4.073
16	.258	.690	1.337	1.746	2.120	2.583	2.921	3.252	3.686	4.015
17	.257	.689	1.333	1.740	2.110	2.567	2.898	3.222	3.646	3.965
18	.257	.688	1.330	1.734	2.101	2.552	2.878	3.197	3.610	3.922
19	.257	.688	1.328	1.729	2.093	2.539	2.861	3.174	3.579	3.883
20	.257	.687	1.325	1.725	2.086	2.528	2.845	3.153	3.552	3.850
21	.257	.686	1.323	1.721	2.080	2.518	2.831	3.135	3.527	3.819
22	.256	.686	1.321	1.717	2.074	2.508	2.819	3.119	3.505	3.792
23	.256	.685	1.319	1.714	2.069	2.500	2.807	3.104	3.485	3.767
24	.256	.685	1.318	1.711	2.064	2.492	2.797	3.091	3.467	3.745
25	.256	.684	1.316	1.708	2.060	2.485	2.787	3.078	3.450	3.725
26	.256	.684	1.315	1.706	2.056	2.479	2.779	3.067	3.435	3.707
27	.256	.684	1.314	1.703	2.052	2.473	2.771	3.057	3.421	3.690
28	.256	.683	1.313	1.701	2.048	2.467	2.763	3.047	3.408	3.674
29	.256	.683	1.311	1.699	2.045	2.462	2.756	3.038	3.396	3.659
30	.256	.683	1.310	1.697	2.042	2.457	2.750	3.030	3.385	3.646
40	.255	.681	1.303	1.684	2.021	2.423	2.704	2.971	3.307	3.551
60	.254	.679	1.296	1.671	2.000	2.390	2.660	2.915	3.232	3.460

11-8 ADEQUACY OF THE REGRESSION MODEL

11-8.1 Residual Analysis



a) Satisfactory

b) Funnel

Variance increases with x

c) double bow

Inequality of variance

d) Nonlinear

Model inadequacy

11-8.2 Coefficient of Determination (R^2)

$$R^2 = \frac{SS_R}{SS_T} = 1 - \frac{SS_E}{SS_T}$$

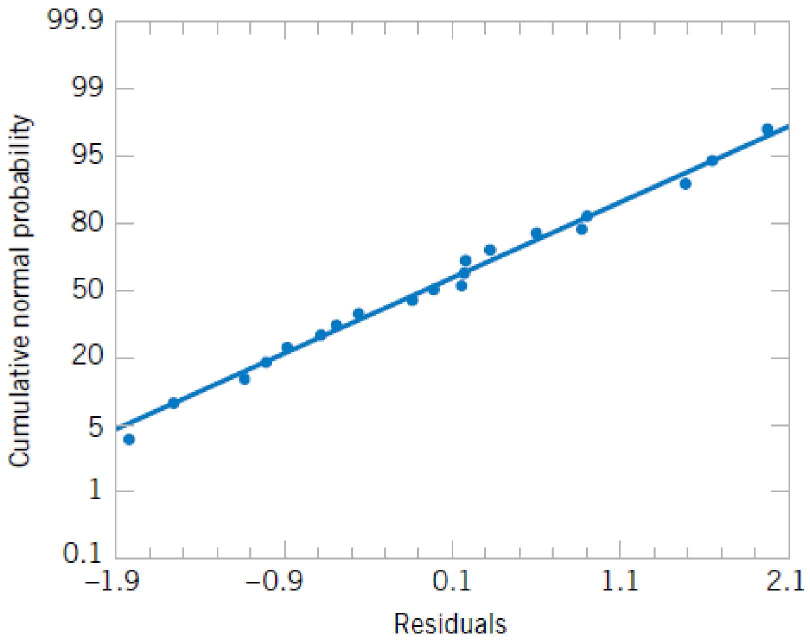


Figure 11-10 Normal probability plot of residuals, Example 11-7.

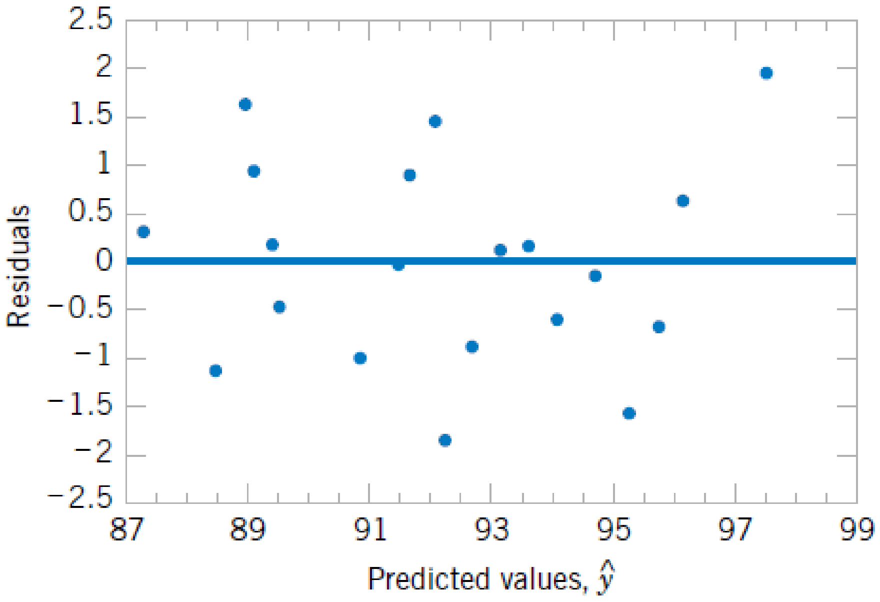


Figure 11-11 Plot of residuals versus predicted oxygen purity $\hat{y}$, Example 11-7.

For the oxygen purity regression model, $R^2 = 0.877$; model accounts for 87.7% of the variability in the data

Adjusted R-Square

$$Adjusted\ R - Square = \frac{\frac{SS_R}{n - K}}{\frac{SS_T}{n - 1}}$$

Multiple Linear Regression

Example 12-1

Data on pull strength of a wire bond in a semiconductor manufacturing process, wire length, and die height to construct an empirical model.

12-1: Multiple Linear Regression Models

Example 12-1

Table 12-2 Wire Bond Data for Example 12-1

Observation Number	Pull Strength y	Wire Length x_1	Die Height x_2	Observation Number	Pull Strength y	Wire Length x_1	Die Height x_2
1	9.95	2	50	14	11.66	2	360
2	24.45	8	110	15	21.65	4	205
3	31.75	11	120	16	17.89	4	400
4	35.00	10	550	17	69.00	20	600
5	25.02	8	295	18	10.30	1	585
6	16.86	4	200	19	34.93	10	540
7	14.38	2	375	20	46.59	15	250
8	9.60	2	52	21	44.88	15	290
9	24.35	9	100	22	54.12	16	510
10	27.50	8	300	23	56.63	17	590
11	17.08	4	412	24	22.13	6	100
12	37.00	11	400	25	21.15	5	400
13	41.95	12	500				

12-1: Multiple Linear Regression Models

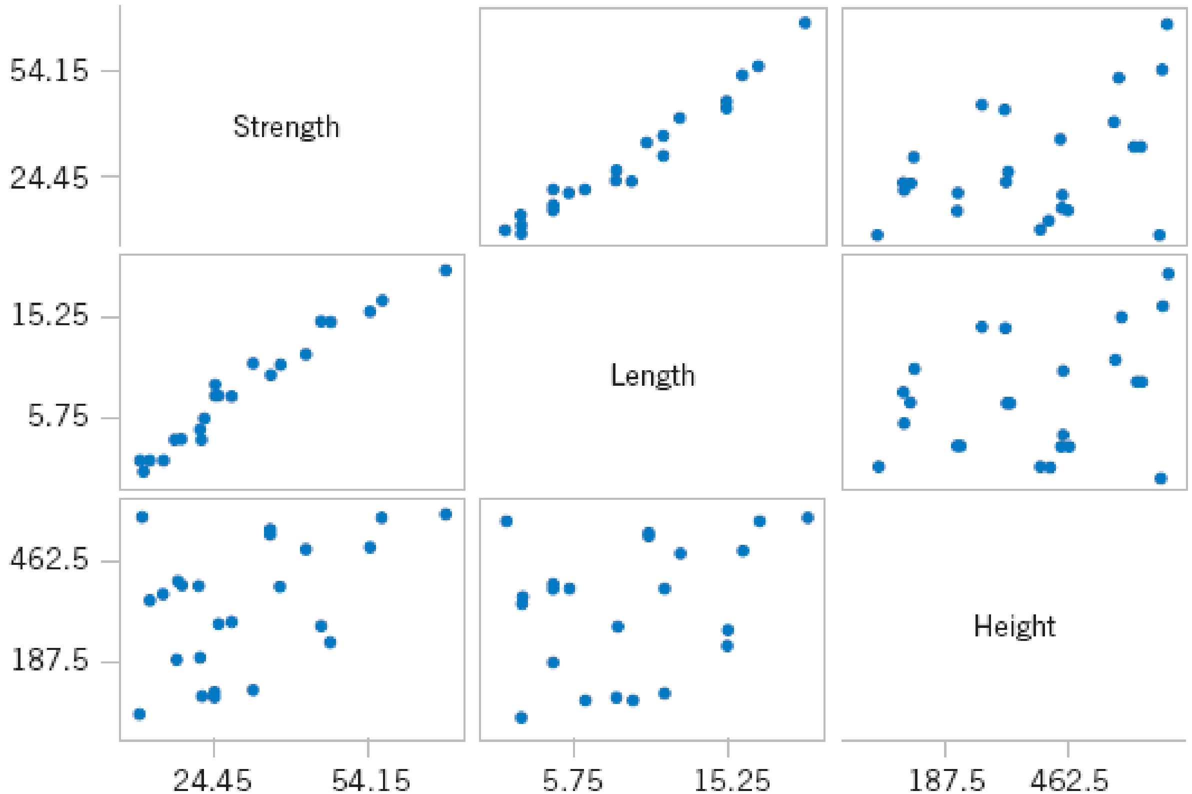


Figure 12-4 Matrix of scatter plots (from Minitab) for the wire bond pull strength data in Table 12-2.

Figure 12-4 Matrix of scatter plots (from Minitab) for the wire bond pull strength data in Table 12-2.

12-1: Multiple Linear Regression Models

Example 12-1

Specifically, we will fit the multiple linear regression model

$$Y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \epsilon$$

where Y = pull strength, x_1 = wire length, and x_2 = die height. From the data in Table 12-2 we calculate

$$n = 25, \sum_{i=1}^{25} y_i = 725.82$$

$$\sum_{i=1}^{25} x_{i1} = 206, \sum_{i=1}^{25} x_{i2} = 8{,}294$$

$$\sum_{i=1}^{25} x_{i1}^2 = 2{,}396, \sum_{i=1}^{25} x_{i2}^2 = 3{,}531{,}848$$

$$\sum_{i=1}^{25} x_{i1}x_{i2} = 77{,}177, \sum_{i=1}^{25} x_{i1}y_i = 8{,}008.47,$$

$$\sum_{i=1}^{25} x_{i2}y_i = 274{,}816.71$$

12-1: Multiple Linear Regression Models

Example 12-1

For the model $Y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \epsilon$, the normal equations 12-10 are

$$n\hat{\beta}_0 + \hat{\beta}_1 \sum_{i=1}^{n} x_{i1} + \hat{\beta}_2 \sum_{i=1}^{n} x_{i2} = \sum_{i=1}^{n} y_i$$

$$\hat{\beta}_0 \sum_{i=1}^{n} x_{i1} + \hat{\beta}_1 \sum_{i=1}^{n} x_{i1}^2 + \hat{\beta}_2 \sum_{i=1}^{n} x_{i1}x_{i2} = \sum_{i=1}^{n} x_{i1}y_i$$

$$\hat{\beta}_0 \sum_{i=1}^{n} x_{i2} + \hat{\beta}_1 \sum_{i=1}^{n} x_{i1}x_{i2} + \hat{\beta}_2 \sum_{i=1}^{n} x_{i2}^2 = \sum_{i=1}^{n} x_{i2}y_i$$

Inserting the computed summations into the normal equations, we obtain

$$25\hat{\beta}_0 + 206\hat{\beta}_1 + 8294\hat{\beta}_2 = 725.82$$

$$206\hat{\beta}_0 + 2396\hat{\beta}_1 + 77{,}177\hat{\beta}_2 = 8{,}008.47$$

$$8294\hat{\beta}_0 + 77{,}177\hat{\beta}_1 + 3{,}531{,}848\hat{\beta}_2 = 274{,}816.71$$

12-1: Multiple Linear Regression Models

Example 12-1

The solution to this set of equations is

$$\hat{\beta}_0 = 2.26379, \quad \hat{\beta}_1 = 2.74427, \quad \hat{\beta}_2 = 0.01253$$

Therefore, the fitted regression equation is

$$\hat{y} = 2.26379 + 2.74427x_1 + 0.01253x_2$$

Practical Interpretation: This equation can be used to predict pull strength for pairs of values of the regressor variables wire length (x_1) and die height (x_2). This is essentially the same regression model given in Section 1-3. Figure 1-16 shows a three-dimensional plot of the plane of predicted values $\hat{y}$ generated from this equation.

Table 12-4 Minitab Multiple Regression Output for the Wire Bond Pull Strength Data

Regression Analysis: Strength versus Length, Height

The regression equation is
Strength = 2.26 + 2.74 Length + 0.0125 Height

Predictor		Coef	SE Coef	T	P	VIF
Constant	$\hat{\beta}_0 \rightarrow$	2.264	1.060	2.14	0.044	
Length	$\hat{\beta}_1 \rightarrow$	2.74427	0.09352	29.34	0.000	1.2
Height	$\hat{\beta}_2 \rightarrow$	0.012528	0.002798	4.48	0.000	1.2

S = 2.288 R-Sq = 98.1% R-Sq (adj) = 97.9%
PRESS = 156.163 R-Sq (pred) = 97.44%

Analysis of Variance

Source	DF	SS	MS	F	P
Regression	2	5990.8	2995.4	572.17	0.000
Residual Error	22	115.2	5.2 $\leftarrow \hat{\sigma}^2$		
Total	24	6105.9			

Source	DF	Seq SS
Length	1	5885.9
Height	1	104.9

Predicted Values for New Observations

New Obs	Fit	SE Fit	95.0% CI	95.0% PI
1	27.663	0.482	(26.663, 28.663)	(22.814, 32.512)

Values of Predictors for New Observations

New Obs	Length	Height
1	8.00	275

12-2.1 Test for Significance of Regression

Table 12-9 Analysis of Variance for Testing Significance of Regression in Multiple Regression

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0
Regression	SS_R	k	MS_R	MS_R/MS_E
Error or residual	SS_E	$n-p$	MS_E	
Total	SS_T	$n-1$		

R^2 and Adjusted R^2

The **coefficient of multiple determination**

$$R^2 = \frac{SS_R}{SS_T} = 1 - \frac{SS_E}{SS_T}$$

- For the wire bond pull strength data, we find that $R^2 = SS_R/SS_T = 5990.7712/6105.9447 = 0.9811$.
- Thus, the model accounts for about 98% of the variability in the pull strength response.

R^2 and Adjusted R^2

The **adjusted R^2** is

$$R^2_{adj} = 1 - \frac{SS_E/(n-p)}{SS_T/(n-1)} \qquad (12\text{-}23)$$

- The adjusted R^2 statistic penalizes the analyst for adding terms to the model.
- It can help guard against overfitting (including regressors that are not really useful)

12-2.2 Tests on Individual Regression Coefficients and Subsets of Coefficients

The hypotheses for testing the significance of any individual regression coefficient:

$$H_0\colon \beta_j = \beta_{j0}$$
$$H_1\colon \beta_j \neq \beta_{j0} \qquad (12\text{-}24)$$

12-2.2 Tests on Individual Regression Coefficients and Subsets of Coefficients

The test statistic is

$$T_0 = \frac{\hat{\beta}_j - \beta_{j0}}{\sqrt{\sigma^2 C_{jj}}} = \frac{\hat{\beta}_j - \beta_{j0}}{se(\hat{\beta}_j)} \qquad (12\text{-}25)$$

- Reject H_0 if $|t_0| > t_{\alpha/2,n-p}$.
- This is called a **partial** or marginal test

Example 12-4

EXAMPLE 12-4 Wire Bond Strength Coefficient Test

Consider the wire bond pull strength data, and suppose that we want to test the hypothesis that the regression coefficient for x_2 (die height) is zero. The hypotheses are

$$H_0\text{: } \beta_2 = 0$$
$$H_1\text{: } \beta_2 \neq 0$$

The main diagonal element of the $(\mathbf{X}'\mathbf{X})^{-1}$ matrix corresponding to $\hat{\beta}_2$ is $C_{22} = 0.0000015$, so the t-statistic in Equation 12-25 is

$$t_0 = \frac{\hat{\beta}_2}{\sqrt{\hat{\sigma}^2 C_{22}}} = \frac{0.01253}{\sqrt{(5.2352)(0.0000015)}} = 4.477$$

12-6: Aspects of Multiple Regression Modeling

Example 12-12

EXAMPLE 12-12 Airplane Sidewall Panels

Sidewall panels for the interior of an airplane are formed in a 1500-ton press. The unit manufacturing cost varies with the production lot size. The data shown below give the average cost per unit (in hundreds of dollars) for this product (y) and the production lot size (x). The scatter diagram, shown in Fig. 12-11, indicates that a second-order polynomial may be appropriate.

y	1.81	1.70	1.65	1.55	1.48	1.40
x	20	25	30	35	40	50
y	1.30	1.26	1.24	1.21	1.20	1.18
x	60	65	70	75	80	90

Example 12-11

Figure 12-11 Data for Example 12-11.

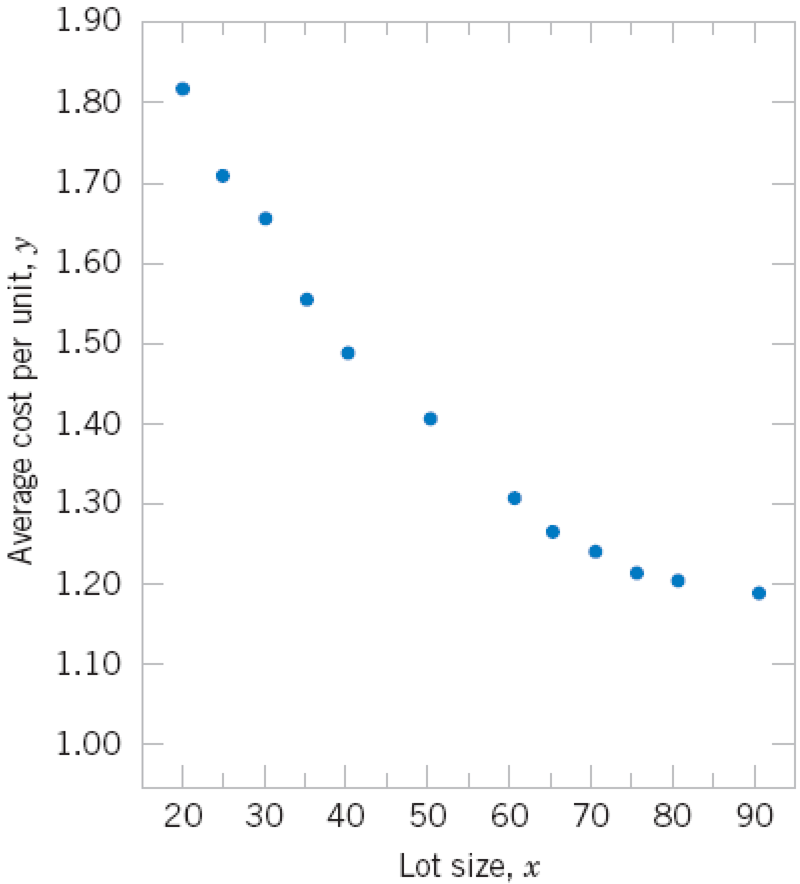


Figure 12-11 Data for Example 12-11.

12-6: Aspects of Multiple Regression Modeling

Example 12-12

Solving the normal equations $\mathbf{X}'\mathbf{X}\hat{\boldsymbol{\beta}} = \mathbf{X}'\mathbf{y}$ gives the fitted model

$$\hat{y} = 2.19826629 - 0.02252236x + 0.00012507x^2$$

Conclusions: The test for significance of regression is shown in Table 12-13. Since $f_0 = 1762.3$ is significant at 1%, we conclude that at least one of the parameters β_1 and β_{11} is not zero. Furthermore, the standard tests for model adequacy do not reveal any unusual behavior, and we would conclude that this is a reasonable model for the sidewall panel cost data.

Table 12-13 Test for Significance of Regression for the Second-Order Model in Example 12-12

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	f_0	P-value
Regression	0.52516	2	0.26258	1762.28	2.12E-12
Error	0.00134	9	0.00015		
Total	0.5265	11			

12-6: Aspects of Multiple Regression Modeling

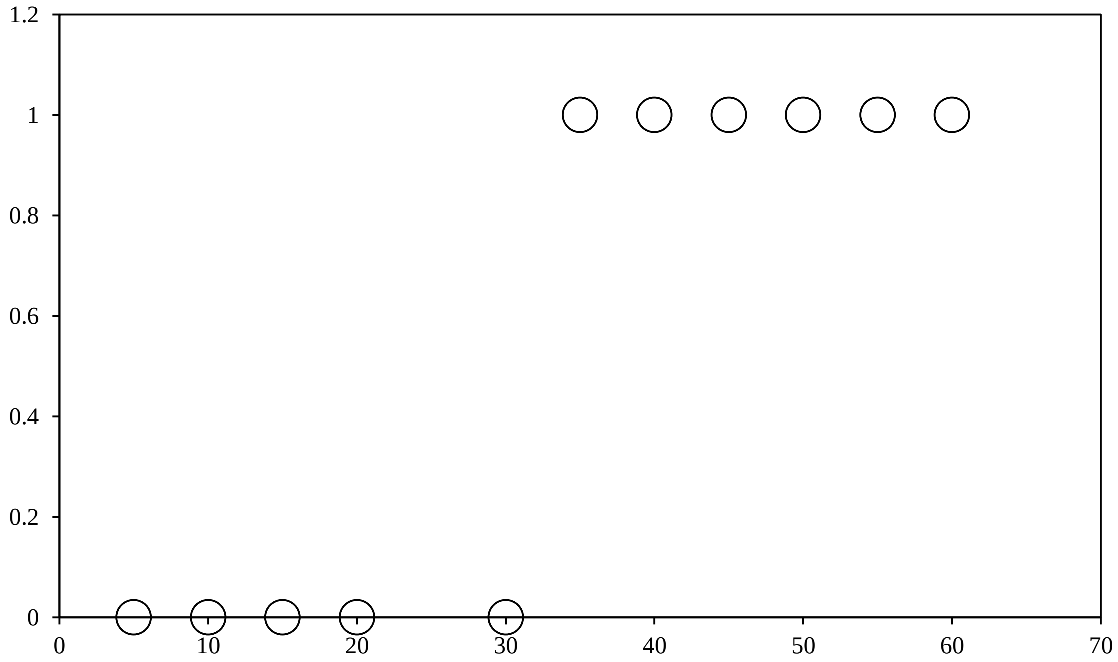
Example 12-13

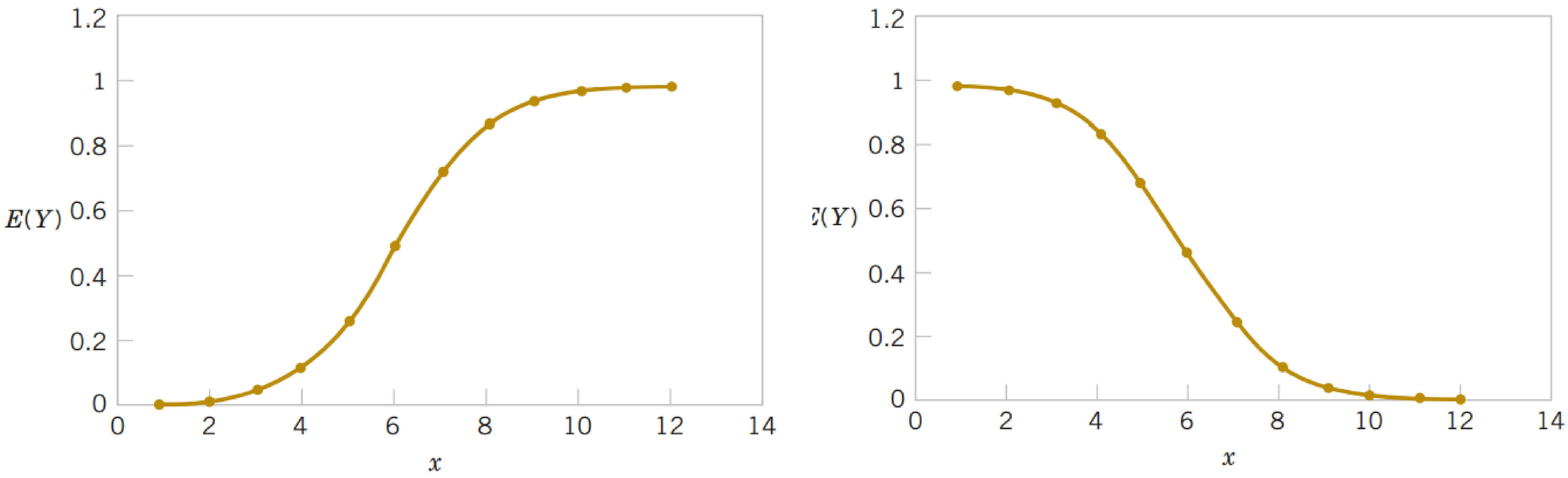
Table 12-16 Analysis of Variance for Example 12-13

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	f_0	P-value
Regression	1012.0595	2	506.0297	1103.69	1.02E-18
$SS_R(\beta_1 \mid \beta_0)$	130.6091	1	130.6091	284.87	4.70E-12
$SS_R(\beta_2 \mid \beta_1, \beta_0)$	881.4504	1	881.4504	1922.52	6.24E-19
Error	7.7943	17	0.4585		
Total	1019.8538	19			

Logistic Regression

- Linear regression often works very well when the response variable is **quantitative.**
- We now consider the situation in which the response variable takes on only two possible values, 0 and 1. These could be arbitrary assignments resulting from observing a **qualitative response.**





1.2
1
0.8
E(Y) 0.6
0.4
0.2
0
0
2
4
6
8
10
12
14
x
1.2
1
0.8
E(Y) 0.6
0.4
0.2
0
0
2
4
6
8
10
12
14
x

Logit response function has the following form

Proportion of 1's (success) at any value of *x*

$$E(Y)=\frac{1}{1+\exp\left[-(\beta_0+\beta_1 x)\right]}$$

$$\frac{E(Y)}{1-E(Y)}=\exp\,(\beta_0+\beta_1 x) \qquad odds = e^{\beta_0+\beta_1 X}$$

The quantity is called the odds. It has a straightforward interpretation: If the odds is 2 for a particular value of *x*, it means that a success is twice as likely as a failure at that value of the regressor *x*.

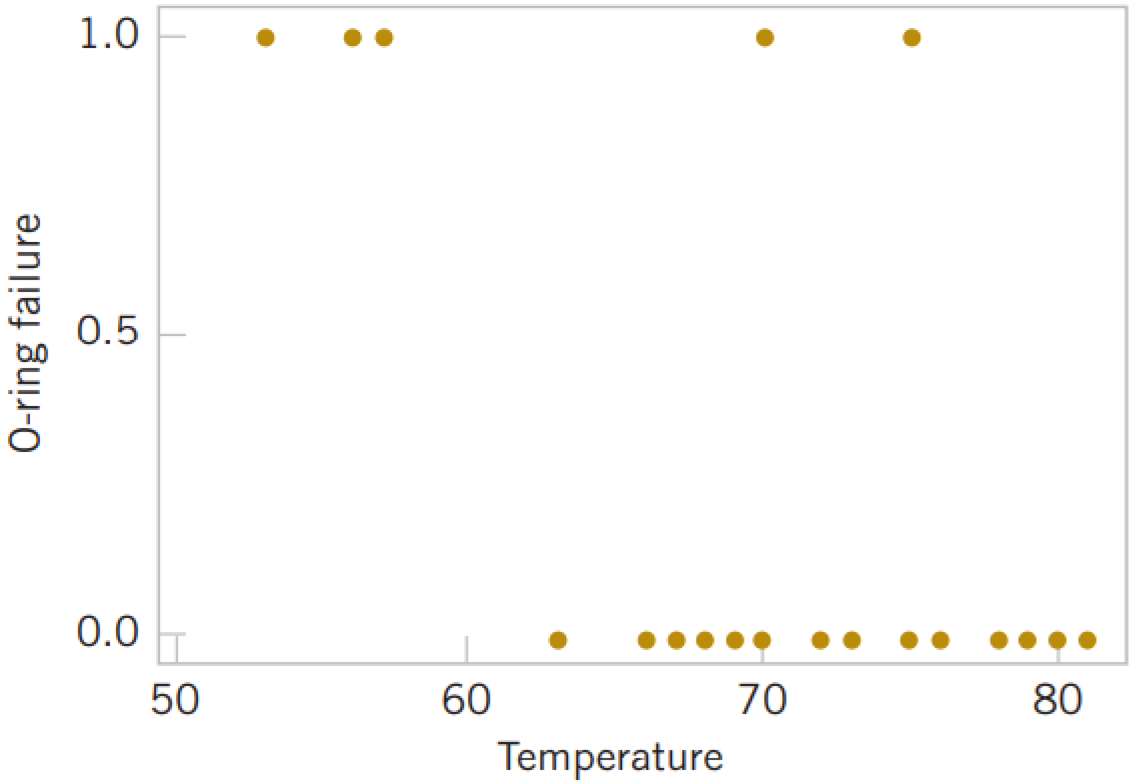
O-ring failure for the space shuttle launches prior to the Challenger disaster of January

Data -shown below- on launch temperature and O-ring failure for the 24 space shuttle launches prior to the *Challenger* disaster of January 1986. Six O-rings were used to seal field joints on the rocket motor assembly. The following table presents the launch temperatures.

Temperature	O-Ring Failure	Temperature	O-Ring Failure	Temperature	O-Ring Failure
53	1	68	0	75	0
56	1	69	0	75	1
57	1	70	0	76	0
63	0	70	1	76	0
66	0	70	1	78	0
67	0	70	1	79	0
67	0	72	0	80	0
67	0	73	0	81	0

"1" in the "O-Ring Failure" column indicates that at least one O-ring failure had occurred on that launch

Figure: Scatter plot of O-ring failures versus launch temperature for 24 space shuttle flights.



Binary Logistic Regression: O-Ring Failure versus Temperature

Link Function: Logit
Response Information

Variable	Value	Count	
O-Ring F	1	7	(Event)
	0	17	
	Total	24	

Logistic Regression Table

Predictor	Coef	SE Coef	Z	P	Odds Ratio	95% Lower	CI Upper
Constant	10.875	5.703	1.91	0.057			
Temperat	–0.17132	0.08344	–2.05	0.040	0.84	0.72	0.99

Log-Likelihood = –11.515
Test that all slopes are zero: G = 5.944, DF = 1, P-Value = 0.015

The odds ratio is 0.84, so every 1 degree increase in temperature reduces the odds of failure by 0.84.

- It is interesting to note that all of these data were available prior to launch.
- However, engineers were unable to effectively analyze the data and use them to provide a convincing argument against launching Challenger to NASA managers.
- Yet a simple regression analysis of the data would have provided a strong quantitative basis for this argument.
- This is one of the more dramatic instances that points out why engineers and scientists need a strong background in basic statistical techniques.

Design and Analysis of Experiments

Statistical design of experiments refers to the process of planning the experiment so that appropriate data will be collected and analyzed by statistical methods, resulting in valid and objective conclusions. The statistical approach to experimental design is necessary if we wish to draw meaningful conclusions from the data.

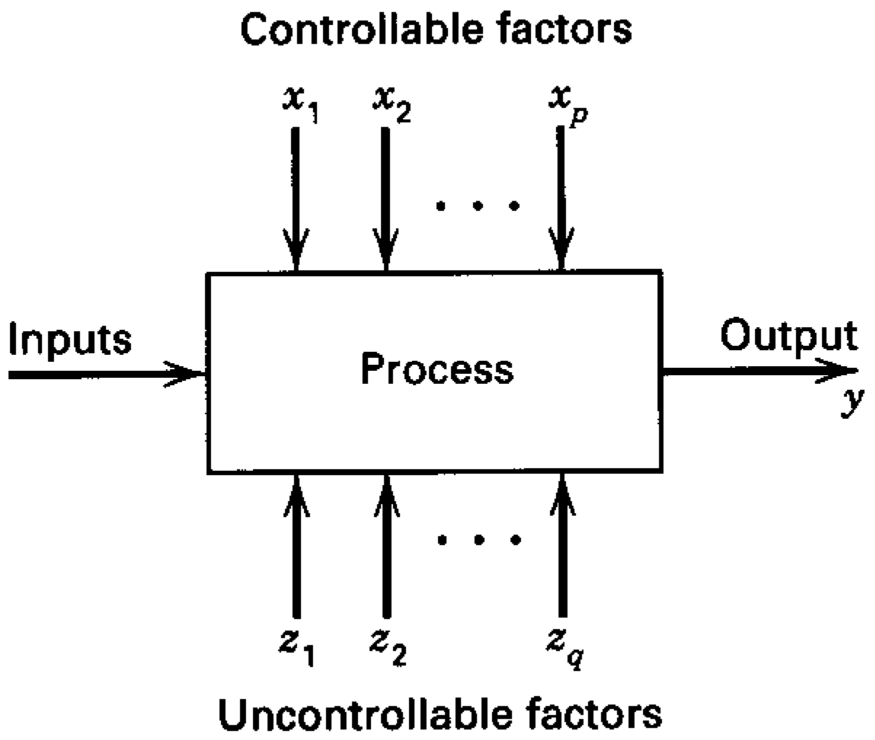
Douglas C. Montgomery

Introduction to Design of Experiments

- An **experiment** is a test or a series of tests
- Statistically based experimental design techniques are used widely in the engineering world
  - New process development
  - New product development
  - Improvement of existing product or process
- "All experiments are designed experiments, some are poorly designed, some are well-designed"

Engineering Designed Experiments

- Reduce **time** to design/develop new products & processes.
- Reduce cost of operation.
- Improve **performance** of existing processes
- Improve **reliability** and performance of products
- Achieve product & process **robustness**
- **Evaluation** of materials, design alternatives, **setting** component & system tolerances, etc.



General model of a process or

Engineering Designed Experiments

Designed experiments are usually employed sequentially.

- ✓ The first experiment with a complex system that has many controllable variables is often a **screening experiment** designed to determine those variables are most important.
- ✓ Subsequent experiments are used to **refine this information** and determine which adjustments to these **critical variables** are required to improve the process.
- ✓ Finally, the objective of the experimenter is **optimization**, that is, to determine those levels of the critical variables that result in the best process performance

Designing Engineering Experiments

Every experiment involves a sequence of activities:

1. **Conjecture** – the original hypothesis that motivates the experiment.
2. **Experiment** – the test performed to investigate the conjecture.
3. **Analysis** – the statistical analysis of the data from the experiment.
4. **Conclusion** – what has been learned about the original conjecture from the experiment. Often the experiment will lead to a revised conjecture, and a new experiment, and so forth.

Strategy of Experimentation

- **“Best-guess” experiments**
  - Used a lot
  - More successful than you might suspect, but there are disadvantages…
- **One-factor-at-a-time (OFAT) experiments**
  - Sometimes associated with the “scientific” or “engineering” method
  - Devastated by interaction, also very inefficient
- **Statistically designed experiments**
  - Based on Fisher’s factorial concept

Planning, Conducting & Analyzing an Experiment

1. Recognition of & statement of problem
2. Choice of factors, levels, and ranges
3. Selection of the response variable(s)
4. Choice of design
5. Conducting the experiment
6. Statistical analysis
7. Drawing conclusions, recommendations

The Basic Principles of DOX

- **Randomization**
    - Running the trials in an experiment in random order
    - Notion of balancing out effects of “lurking” variables
- **Replication**
    - Sample size (improving precision of effect estimation, estimation of error or background noise)
    - Replication versus repeat measurements?
- **Blocking**
    - Dealing with nuisance factors

- **Fixed-Effects Model**
- **Random-Effects Model**
- **Randomized Complete Block Design**

Completely Randomized Single-Factor Experiment

<u>Example</u>: A manufacturer of paper used for making grocery bags is interested in improving the tensile strength of the product. Product engineering thinks that tensile strength is a function of the hardwood concentration in the pulp and that the range of hardwood concentrations of practical interest is between 5 and 20%. A team of engineers responsible for the study decides to investigate ***four levels*** of hardwood concentration: 5%, 10%, 15%, and 20%. They decide to make up ***six test specimens at each concentration level***, using a pilot plant. All 24 specimens are tested on a laboratory tensile tester, ***in random order***. The data from this experiment are shown below.

Table 13.1: Tensile strength of paper (psi)

Hardwood Concentration (%)	Observations						Totals	Averages
	1	2	3	4	5	6		
5	7	8	15	11	9	10	60	10.00
10	12	17	13	18	19	15	94	15.67
15	14	18	19	17	16	18	102	17.00
20	19	25	22	23	18	20	127	21.17
							383	15.96

- The levels of the factor are sometimes called **treatments**.
- Each treatment has six observations or **replicates**.
- The runs are run in **random** order.

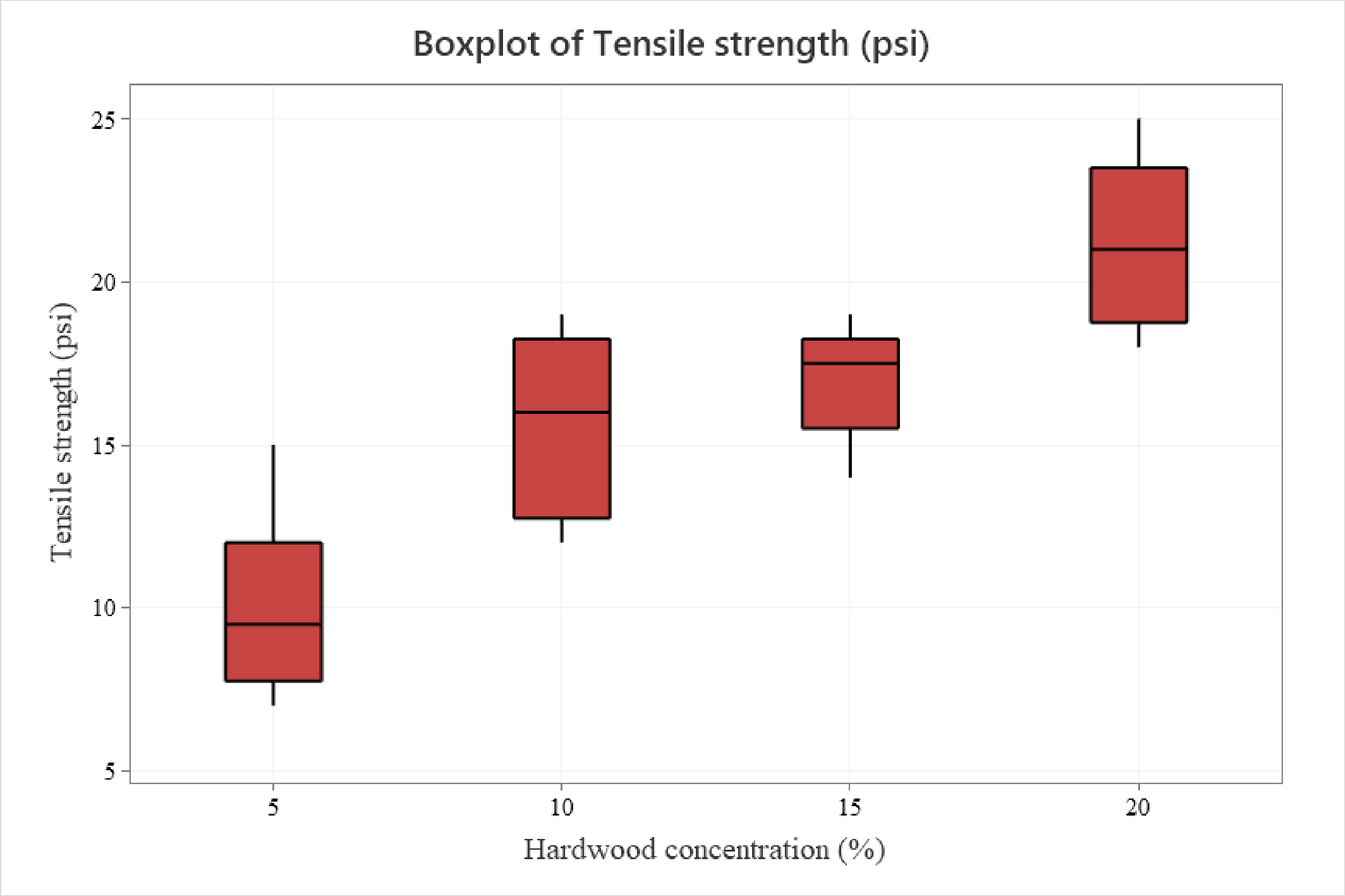
Why Randomization?

- By randomizing the order of the 24 runs, the effect of any nuisance variable that may influence the observed tensile strength is approximately balanced out

- For example, suppose that there is a **warm-up effect** on the tensile testing machine; that is, the longer the machine is on, the greater the observed tensile strength. If all 24 runs are made in order of increasing hardwood concentration (that is, all six 5% concentration specimens are tested first, followed by all six 10% concentration specimens, etc.), *any observed differences in tensile strength could also be due to the warm-up effect*

- When statistical significance is observed in a **randomized experiment**, the *experimenter can be confident in the conclusion that the difference in treatments resulted in the difference in response*. That is, we can be confident that a cause-and-effect relationship has been found.

Box plots show the variability of the observations within a treatment (factor level) and the variability between the treatments.



Changing the hardwood concentration has an effect on tensile strength

The Analysis of Variance

- ✓ Suppose there are ***a*** different levels of a single factor that we wish to compare. The levels are sometimes called **treatments**.

- ✓ We may describe the observations by the linear statistical model:

$$Y_{ij} = \mu + \tau_i + \epsilon_{ij} \begin{cases} i = 1, 2, \ldots, a \\ j = 1, 2, \ldots, n \end{cases}$$

μ is a parameter common to all treatments called the **overall mean**
$\boldsymbol{\tau_i}$ is a parameter associated with the *i*th treatment called the ***i*th treatment effect**
$\boldsymbol{\varepsilon}_{ij}$ is a random error component

- ✓ The model could be written as

$$Y_{ij} = \mu_i + \epsilon_{ij} \begin{cases} i = 1, 2, \ldots, a \\ j = 1, 2, \ldots, n \end{cases}$$

$$Y_{ij} = \mu_i + \epsilon_{ij} \begin{cases} i = 1, 2, \ldots, a \\ j = 1, 2, \ldots, n \end{cases}$$

where $\mu_i = \mu + \tau_i$ is the mean of the *i*th treatment.

General Notes:

- Each treatment defines a population that has mean μ_i consisting of the overall mean μ plus an effect τ_i
- The errors ε_{ij} are normally and independently distributed with mean zero and variance σ^2. Therefore, each treatment can be thought of as a normal population with mean μ_i and variance σ^2

TABLE • 13-2 Typical Data for a Single-Factor Experiment

Treatment	Observations				Totals	Averages
1	y_{11}	y_{12}	$\cdots$	y_{1n}	$y_{1\cdot}$	$\bar{y}_{1\cdot}$
2	y_{21}	y_{22}	$\cdots$	y_{2n}	$y_{2\cdot}$	$\bar{y}_{2\cdot}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
a	y_{a1}	y_{a2}	$\cdots$	y_{an}	$y_{a\cdot}$	$\bar{y}_{a\cdot}$
					$y_{\cdot\cdot}$	$\bar{y}_{1\cdot\cdot}$

- The observations are taken in random order and that the environment (often called the experimental units) in which the treatments are used is as uniform as possible, this experimental design is called a **completely randomized design (CRD).**

- First, the experimenter could have specifically chosen the $\boldsymbol{a}$ treatments. In this situation, we wish to test hypotheses about the treatment means, and conclusions cannot be extended to similar treatments that were not considered. In addition, we may wish to estimate the treatment effects. This is called the **fixed-effects model**.

- Alternatively, the $\boldsymbol{a}$ treatments could be a random sample from a larger population of treatments. In this situation, we would like to be able to extend the conclusions (which are based on the sample of treatments) to all treatments in the population whether or not they were explicitly considered in the experiment. Here the treatment effects $\boldsymbol{\tau_i}$ are random variables, and knowledge about the particular ones investigated is relatively unimportant. Instead, we test hypotheses about the variability of the $\boldsymbol{\tau_i}$ and try to estimate this variability. This is called the **random-effects, or components of variance model**.

$$Y_{ij} = \mu + \tau_i + \epsilon_{ij} \begin{cases} i = 1, 2, \ldots, a \\ j = 1, 2, \ldots, n \end{cases}$$

Model for a single-factor experiment.

Fixed-effects model.

- In the fixed-effects model, the treatment effects $\boldsymbol{\tau_i}$ are usually defined as deviations from the overall mean, so that

$$\sum_{i=1}^{a} \tau_i = 0$$

Typical Data for a Single-Factor Experiment

Treatment	Observations				Totals	Averages
1	y_{11}	y_{12}	$\cdots$	y_{1n}	$y_{1\cdot}$	$\bar{y}_{1\cdot}$
2	y_{21}	y_{22}	$\cdots$	y_{2n}	$y_{2\cdot}$	$\bar{y}_{2\cdot}$
$\vdots$	$\vdots$	$\vdots$	$\vdots\vdots\vdots$	$\vdots$	$\vdots$	$\vdots$
a	y_{a1}	y_{a2}	$\cdots$	y_{an}	$y_{a\cdot}$	$\bar{y}_{a\cdot}$
					$y_{\cdot\cdot}$	$\bar{y}_{\cdot\cdot}$

$$y_{i\cdot} = \sum_{j=1}^{n} y_{ij} \qquad \bar{y}_{i\cdot} = y_{i\cdot}/n \qquad i = 1, 2, \ldots, a$$

$$y_{\cdot\cdot} = \sum_{i=1}^{a} \sum_{j=1}^{n} y_{ij} \qquad \bar{y}_{\cdot\cdot} = y_{\cdot\cdot}/N$$

$N = an$ is the total number of observations.

$\boldsymbol{yi.}$ *represnt the total of the observation under the ith treatment*

$\bar{\boldsymbol{y}}\boldsymbol{i.}$ *represnt the average of the observations under the ith treatment*

$\boldsymbol{y..}$ *represnt the grand of the observations*

$\bar{\boldsymbol{y}}\boldsymbol{..}$ *represnt the grand mean of all observations.*

We are interested in testing the equality of the ***a*** treatment means $\mu_1, \mu_2, \ldots, \mu_a$. We find that this is equivalent to testing the hypotheses

$$H_0\colon \tau_1 = \tau_2 = \cdots = \tau_a = 0$$
$$H_1\colon \tau_i \neq 0 \quad \text{for at least one } i$$

$$H_0: \tau_1 = \tau_2 = \cdots = \tau_a = 0$$
$$H_1: \tau_i \neq 0 \quad \text{for at least one } i$$

Thus, if the null hypothesis is true, each observation consists of the overall mean μ plus a realization of the random error component $\boldsymbol{\varepsilon_{ij}}$. This is equivalent to saying that all ***N*** observations are taken from a normal distribution with mean $\boldsymbol{\mu}$ and variance $\boldsymbol{\sigma^2}$. Therefore, if the null hypothesis is true, changing the levels of the factor has no effect on the mean response.

The total variability in the data is described by the total sum of square

$$SS_T = \sum_{i=1}^{a} \sum_{j=1}^{n} (y_{ij} - \bar{y}_{..})^2$$

The **sum of squares identity** is

$$\sum_{i=1}^{a} \sum_{j=1}^{n} (y_{ij} - \bar{y}_{..})^2 = n \sum_{i=1}^{a} (\bar{y}_{i.} - \bar{y}_{..})^2 + \sum_{i=1}^{a} \sum_{j=1}^{n} (y_{ij} - \bar{y}_{i.})^2$$

or symbolically

$$SS_T = SS_{\text{Treatments}} + SS_E$$

The expected value of the treatment sum of squares is

$$E(SS_{\text{Treatments}}) = (a-1)\sigma^2 + n\sum_{i=1}^{a}\tau_i^2$$

and the expected value of the error sum of squares is

$$E(SS_E) = a(n-1)\sigma^2$$

$$an - 1 = a - 1 + a(n-1)$$

The ratio

$$MS_{\text{Treatments}} = SS_{\text{Treatments}}/(a-1)$$

is called the **mean square for treatments.** Now if the null hypothesis H_0: $\tau_1 = \tau_2 = \cdots = \tau_a = 0$ is true, $MS_{\text{Treatments}}$ is an unbiased estimator of σ^2 because $\sum_{i=1}^{a}\tau_i = 0$. However, if H_1 is true, $MS_{\text{Treatments}}$ estimates σ^2 plus a positive term that incorporates variation due to the systematic difference in treatment means.

$$MS_E = SS_E/[a(n-1)]$$

is an unbiased estimator of σ^2 regardless of whether or not H_0 is true. We can also show that $MS_{\text{Treatments}}$ and MS_E are independent. Consequently, we can show that if the null hypothesis H_0 is true, the ratio

$$F_0 = \frac{SS_{\text{Treatments}}/(a-1)}{SS_E/[a(n-1)]} = \frac{MS_{\text{Treatments}}}{MS_E}$$

$$H_0: \tau_1 = \tau_2 = \cdots = \tau_a = 0$$
$$H_1: \tau_i \neq 0 \quad \text{for at least one } i$$

Reject H_0 if F_0 is greater than $f_{\alpha,\, a-1,\, a(n-1)}$

$$SS_T = \sum_{i=1}^{a} \sum_{j=1}^{n} y_{ij}^2 - \frac{y_{..}^2}{N}$$

and

$$SS_{\text{Treatments}} = \sum_{i=1}^{a} \frac{y_{i.}^2}{n} - \frac{y_{..}^2}{N}$$

The error sum of squares is obtained by subtraction as

$$SS_E = SS_T - SS_{\text{Treatments}}$$

The analysis of Variance for a single-Factor Experiment, Fixed-Effect Model

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0
Treatments	$SS_{\text{Treatments}}$	$a - 1$	$MS_{\text{Treatments}}$	$\frac{MS_{\text{Treatments}}}{MS_E}$
Error	SS_E	$a(n - 1)$	MS_E	
Total	SS_T	$an - 1$		

Table 13.1: Tensile strength of paper (psi)

Hardwood Concentration (%)	Observations 1	2	3	4	5	6	Totals	Averages
5	7	8	15	11	9	10	60	10.00
10	12	17	13	18	19	15	94	15.67
15	14	18	19	17	16	18	102	17.00
20	19	25	22	23	18	20	127	21.17
							383	15.96

$H_0: \tau_1 = \tau_2 = \tau_3 = \tau_4 = 0$ $\qquad$ $H_1: \tau_i \neq 0$ for at least one i

$\alpha = 0.01$

$$SS_T = \sum_{i=1}^{4} \sum_{j=1}^{6} y_{ij}^2 - \frac{y_{..}^2}{N}$$

$$= (7)^2 + (8)^2 + \cdots + (20)^2 - \frac{(383)^2}{24} = 512.96$$

$$SS_{\text{Treatments}} = \sum_{i=1}^{4} \frac{y_{i.}^2}{n} - \frac{y_{..}^2}{N}$$

$$= \frac{(60)^2 + (94)^2 + (102)^2 + (127)^2}{6} - \frac{(383)^2}{24} = 382.79$$

$$SS_E = SS_T - SS_{\text{Treatments}}$$

$$= 512.96 - 382.79 = 130.17$$

$$f_{\alpha,a-1,a(n-1)} = f_{0.01,3,20} = 4.94\ (From\ Table)$$

TABLE • 13-4 ANOVA for the Tensile Strength Data

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	f_0	*P*-value
Hardwood					
concentration	382.79	3	127.60	19.60	3.59 E-6
Error	130.17	20	6.51		
Total	512.96	23			

Confidence Interval on a Treatment Mean

A 100(1 – α)% confidence interval on the mean of the ith treatment μ_i is

$$\bar{y}_{i\cdot} - t_{\alpha/2,a(n-1)}\sqrt{\frac{MS_E}{n}} \le \mu_i \le \bar{y}_{i\cdot} + t_{\alpha/2,a(n-1)}\sqrt{\frac{MS_E}{n}} \qquad (13\text{-}11)$$

95% CI

$$[\bar{y}_{4\cdot} \pm t_{0.025,20}\sqrt{MS_E/n}]$$

$$[21.167 \pm (2.086)\sqrt{6.51/6}]$$

$$19.00 \text{ psi} \le \mu_4 \le 23.34 \text{ psi}$$

Confidence Interval on a Difference in Treatment Means

A $100(1-\alpha)$ percent confidence interval on the difference in two treatment means $\mu_i - \mu_j$ is

$$\bar{y}_{i\cdot} - \bar{y}_{j\cdot} - t_{\alpha/2,a(n-1)}\sqrt{\frac{2MS_E}{n}} \le \mu_i - \mu_j \le \bar{y}_{i\cdot} - \bar{y}_{j\cdot} + t_{\alpha/2,a(n-1)}\sqrt{\frac{2MS_E}{n}} \qquad (13\text{-}12)$$

A 95% CI on the difference in means $\mu_3 - \mu_2$ is computed from Equation 13-12 as follows:

$$\left[\bar{y}_{3\cdot} - \bar{y}_{2\cdot} \pm t_{0.025,20}\sqrt{2MS_E/n}\right]$$

$$\left[17.00 - 15.67 \pm (2.086)\sqrt{2(6.51)/6}\right]$$

or

$$-1.74 \le \mu_3 - \mu_2 \le 4.40$$

13-1. Consider the following computer output.

Source	DF	SS	MS	F	*P*-value
Factor	?	117.4	39.1	?	?
Error	16	396.8	?		
Total	19	514.2			

(a) How many levels of the factor were used in this experiment?
(b) How many replicates did the experimenter use?
(c) Fill in the missing information in the ANOVA table. Use bounds for the *P*-value.
(d) What conclusions can you draw about differences in the factor-level means?

13-2. Consider the following computer output for an experiment. The factor was tested over four levels.

Source	DF	SS	MS	F	*P*-value
Factor	?	?	330.4716	4.42	?
Error	?	?	?		
Total	31	?			

(a) How many replicates did the experimenter use?

(b) Fill in the missing information in the ANOVA table. Use bounds for the *P*-value.

(c) What conclusions can you draw about differences in the factor-level means?

Unbalanced Experiment

In some single-factor experiments, the number of observations taken under each treatment may be different. We then say that the design is **unbalanced**. In this situation, slight modifications must be made in the sums of squares formulas. Let n_i observations be taken under treatment i ($i = 1, 2, \ldots, a$), and let the total number of observations $N = \sum_{i=1}^{a} n_i$. The computational formulas for SS_T and $SS_{\text{Treatments}}$ are as shown in the following definition.

Computing Formulas for ANOVA: Single Factor with Unequal Sample Sizes

The sums of squares computing formulas for the ANOVA with unequal sample sizes n_i in each treatment are

$$SS_T = \sum_{i=1}^{a} \sum_{j=1}^{n_i} y_{ij}^2 - \frac{y_{..}^2}{N} \qquad (13\text{-}13)$$

$$SS_{\text{Treatments}} = \sum_{i=1}^{a} \frac{y_{i.}^2}{n_i} - \frac{y_{..}^2}{N} \qquad (13\text{-}14)$$

and

$$SS_E = SS_T - SS_{\text{Treatments}} \qquad (13\text{-}15)$$

MULTIPLE COMPARISONS FOLLOWING THE ANOVA

When the null hypothesis $H_0: \tau_1 = \tau_2 = \cdots = \tau_a = 0$ is rejected in the ANOVA, we know that some of the treatment or factor-level means are different. However, the ANOVA does not identify which means are different. Methods for investigating this issue are called **multiple comparisons methods**.

Fisher's least significant difference (LSD) method

The Fisher LSD method compares all pairs of means with the null hypotheses $H_0: \mu_i = \mu_j$ (for all $i \neq j$) using the t-statistic

$$t_0 = \frac{\bar{y}_{i\cdot} - \bar{y}_{j\cdot}}{\sqrt{\dfrac{2MS_E}{n}}}$$

Assuming a two-sided alternative hypothesis, the pair of means μ_i and μ_j would be declared significantly different if

$$|\bar{y}_{i\cdot} - \bar{y}_{j\cdot}| > \text{LSD}$$

where LSD, the least significant difference, is

Least Significant Difference for Multiple Comparisons

$$\text{LSD} = t_{\alpha/2,a(n-1)} \sqrt{\frac{2MS_E}{n}} \qquad (13\text{-}16)$$

If the sample sizes are different in each treatment, the LSD is defined as

$$\text{LSD} = t_{\alpha/2,N-a} \sqrt{MS_E\left(\frac{1}{n_i} + \frac{1}{n_j}\right)}$$

Fisher's least significant difference (LSD) method

Example 13-2 We apply the Fisher LSD method to the hardwood concentration experiment. There are $a = 4$ means, $n = 6$, $MS_E = 6.51$, and $t_{0.025,20} = 2.086$. The treatment means are

$$\bar{y}_{1\cdot} = 10.00 \text{ psi} \qquad \bar{y}_{2\cdot} = 15.67 \text{ psi}$$
$$\bar{y}_{3\cdot} = 17.00 \text{ psi} \qquad \bar{y}_{4\cdot} = 21.17 \text{ psi}$$

The value of LSD is $\text{LSD} = t_{0.025,20}\sqrt{2MS_E / n} = 2.086\sqrt{2(6.51)/6} = 3.07$. Therefore, any pair of treatment averages that differs by more than 3.07 implies that the corresponding pair of treatment means are different.

The comparisons among the observed treatment averages are as follows:

$$4 \text{ vs. } 1 = 21.17 - 10.00 = 11.17 > 3.07$$
$$4 \text{ vs. } 2 = 21.17 - 15.67 = 5.50 > 3.07$$
$$4 \text{ vs. } 3 = 21.17 - 17.00 = 4.17 > 3.07$$
$$3 \text{ vs. } 1 = 17.00 - 10.00 = 7.00 > 3.07$$
$$3 \text{ vs. } 2 = 17.00 - 15.67 = 1.33 < 3.07$$
$$2 \text{ vs. } 1 = 15.67 - 10.00 = 5.67 > 3.07$$

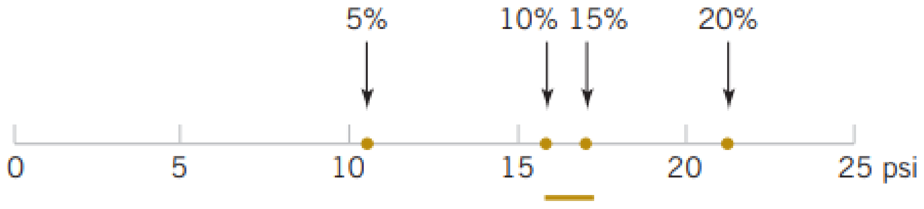


FIGURE 13-2 Results of Fisher's LSD method in Example 13-2.

RESIDUAL ANALYSIS AND MODEL CHECKING

- The normality assumption can be checked by constructing a normal probability plot of the residuals

- To check the assumption of equal variances at each factor level, plot the residuals against the factor levels and compare the spread in the residuals

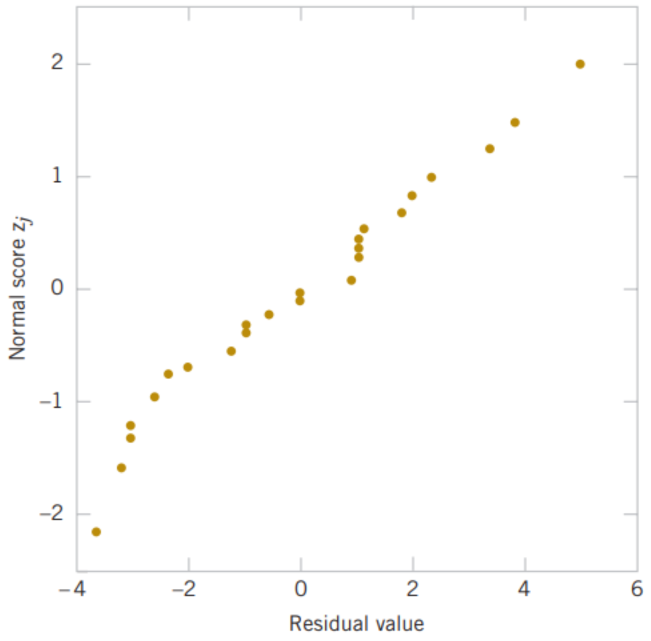


FIGURE 13-4 Normal probability plot of residuals from the hardwood concentration experiment.

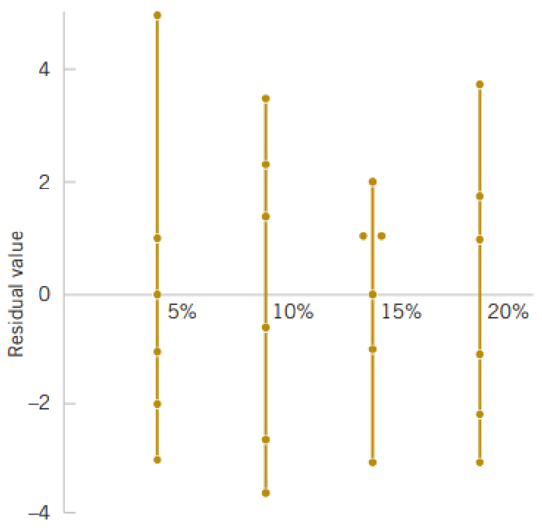


FIGURE 13-5 Plot of residuals versus factor levels (hardwood concentration).

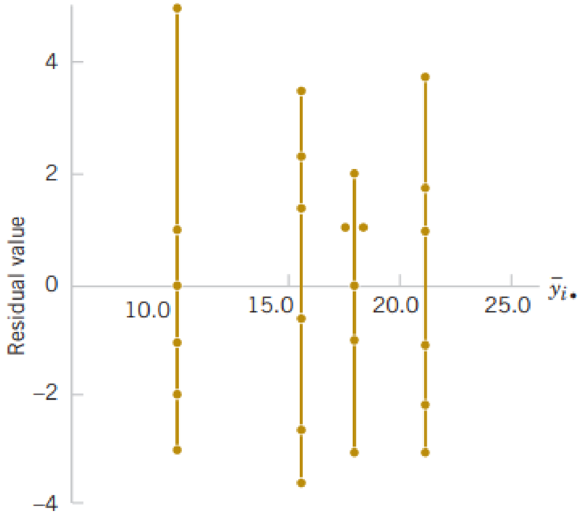


FIGURE 13-6 Plot of residuals versus $\bar{y}_i$.

Problem 13-6. In "Orthogonal Design for Process Optimization and Its Application to Plasma Etching" (Solid State Technology, May 1987), G. Z. Yin and D. W. Jillie described an experiment to determine the effect of C_2F_6 flow rate on the uniformity of the etch on a silicon wafer used in integrated circuit manufacturing. Three flow rates are used in the experiment, and the resulting uniformity (in percent) for six replicates follows.

C_2F_6 Flow (SCCM)	Observations					
	1	2	3	4	5	6
125	2.7	4.6	2.6	3.0	3.2	3.8
160	4.9	4.6	5.0	4.2	3.6	4.2
200	4.6	3.4	2.9	3.5	4.1	5.1

(a) Does C_2F_6 flow rate affect etch uniformity? Construct box plots to compare the factor levels and perform the analysis of variance. Use $\alpha = 0.05$.

(b) Do the residuals indicate any problems with the underlying assumptions?

Problem 13-7. The compressive strength of concrete is being studied, and four different mixing techniques are being investigated. The following data have been collected.

Mixing Technique	Compressive Strength (psi)			
1	3129	3000	2865	2890
2	3200	3300	2975	3150
3	2800	2900	2985	3050
4	2600	2700	2600	2765

(a) Test the hypothesis that mixing techniques affect the strength of the concrete. Use $\alpha = 0.05$.
(b) Find the P-value for the F-statistic computed in part (a).
(c) Analyze the residuals from this experiment.

The Random-Effects Model

- In many situations, the factor of interest has a large number of possible levels. The analyst is interested in drawing conclusions about *the entire population* of factor levels. If the experimenter randomly selects a of these levels from the population of factor levels, we say that the *factor is a random factor*. Because the levels of the factor actually used in the experiment are chosen randomly, the conclusions reached ***are valid for the entire population of factor levels***.

- Notice that this is a very different situation than the one we encountered in the fixed-effects case in which the ***conclusions apply only for the factor levels used in the experiment***.

ANOVA AND VARIANCE COMPONENTS

The linear statistical model is

$$Y_{ij} = \mu + \tau_i + \epsilon_{ij} \begin{cases} i = 1, 2, \ldots, a \\ j = 1, 2, \ldots, n \end{cases} \tag{13-18}$$

where the treatment effects τ_i and the errors ε_{ij} are independent random variables. Note that the model is identical in structure to the fixed-effects case, but the parameters have a different interpretation. If the variance of the treatment effects τ_i is σ_τ^2, by independence the variance of the response is

$$V(Y_{ij}) = \sigma_\tau^2 + \sigma^2 \tag{13-19}$$

The variances σ_τ^2 and σ^2 are called **variance components**, and the model, Equation 13-19, is called the **components of variance model** or the **random-effects model**. To test hypotheses in this model, we assume that the errors ϵ_{ij} are normally and independently distributed with mean zero and variance σ^2 and that the treatment effects τ_i are normally and independently distributed with mean zero and variance σ_τ^2.*

*The assumption that the $\{\tau_i\}$ are independent random variables implies that the usual assumption of $\sum_{i=1}^{n} \tau_i = 0$ from the fixed-effects model does not apply to the random-effects model.

ANOVA AND VARIANCE COMPONENTS

For the random-effects model, testing the hypothesis that the individual treatment effects are zero is meaningless. It is more appropriate to test hypotheses about σ_τ^2. Specifically,

$$H_0: \sigma_\tau^2 = 0 \qquad H_1: \sigma_\tau^2 > 0$$

If $\sigma_\tau^2 = 0$, all treatments are identical; but if $\sigma_\tau^2 > 0$, there is variability between treatments.

The ANOVA decomposition of total variability is still valid; that is,

$$SS_T = SS_{\text{Treatments}} + SS_E \tag{13-20}$$

However, the expected values of the mean squares for treatments and error are somewhat different than in the fixed-effects case.

Expected Values of Mean Squares: Random Effects

In the random-effects model for a single-factor, completely randomized experiment, the expected mean square for treatments is

$$E(MS_{\text{Treatments}}) = E\left(\frac{SS_{\text{Treatments}}}{a-1}\right) = \sigma^2 + n\sigma_\tau^2 \tag{13-21}$$

and the expected mean square for error is

$$E(MS_E) = E\left[\frac{SS_E}{a(n-1)}\right] = \sigma^2 \tag{13-22}$$

$$H_0: \sigma_\tau^2 = 0 \qquad H_1: \sigma_\tau^2 > 0 \qquad F_0 = \frac{MS_{\text{Treatments}}}{MS_E}$$

- ✓ The null hypothesis would be rejected at the *a*-level of significance if the computed value of the test statistic $f_0 > f_{\alpha, a-1, a(n-1)}$

- ✓ The computational procedure and construction of the ANOVA table for the random-effects model are identical to the fixed-effects case. The conclusions, however, are quite different because they apply to the entire population of treatments.

- ✓ **Normality assumption on the observations is not required**.

$$\hat{\sigma}^2 = MS_E \qquad (13\text{-}24)$$

and

$$\hat{\sigma}_\tau^2 = \frac{MS_{\text{Treatments}} - MS_E}{n} \qquad (13\text{-}25)$$

Example 13-4 **Textile Manufacturing** In *Design and Analysis of Experiments*, 8th edition (John Wiley, 2012), D. C. Montgomery describes a single-factor experiment involving the random-effects model in which a textile manufacturing company weaves a fabric on a large number of looms. The company is interested in loom-to-loom variability in tensile strength. To investigate this variability, a manufacturing engineer selects four looms at random and makes four strength determinations on fabric samples chosen at random from each loom. The data are shown in Table 13-7 and the ANOVA is summarized in Table 13-8.

TABLE • 13-7 Strength Data for Example 13-4

	Observations					
Loom	**1**	**2**	**3**	**4**	**Total**	**Average**
1	98	97	99	96	390	97.5
2	91	90	93	92	366	91.5
3	96	95	97	95	383	95.8
4	95	96	99	98	388	97.0
					1527	95.45

TABLE • 13-8 Analysis of Variance for the Strength Data

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	f_0	*P*-value
Looms	89.19	3	29.73	15.68	1.88 E-4
Error	22.75	12	1.90		
Total	111.94	15			

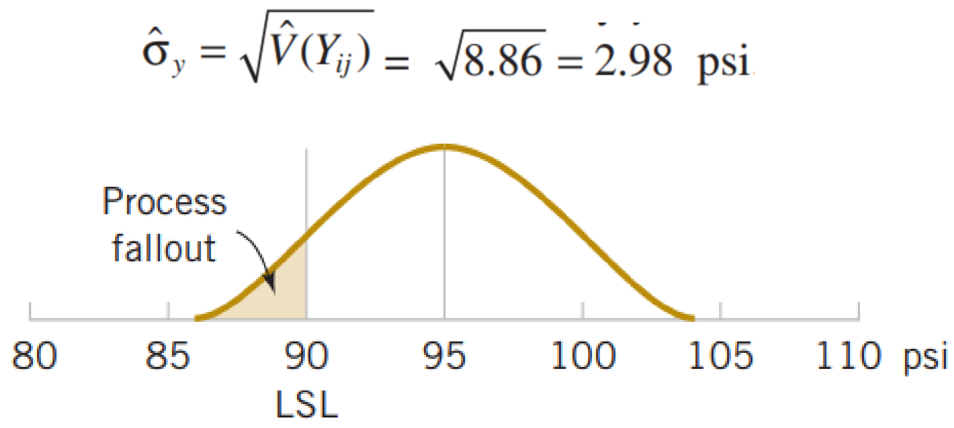
From the analysis of variance, we conclude that the looms in the plant differ significantly in their ability to produce fabric of uniform strength. The variance components are estimated by $\hat{\sigma}^2 = 1.90$ and

$$\hat{\sigma}_\tau^2 = \frac{29.73 - 1.90}{4} = 6.96$$

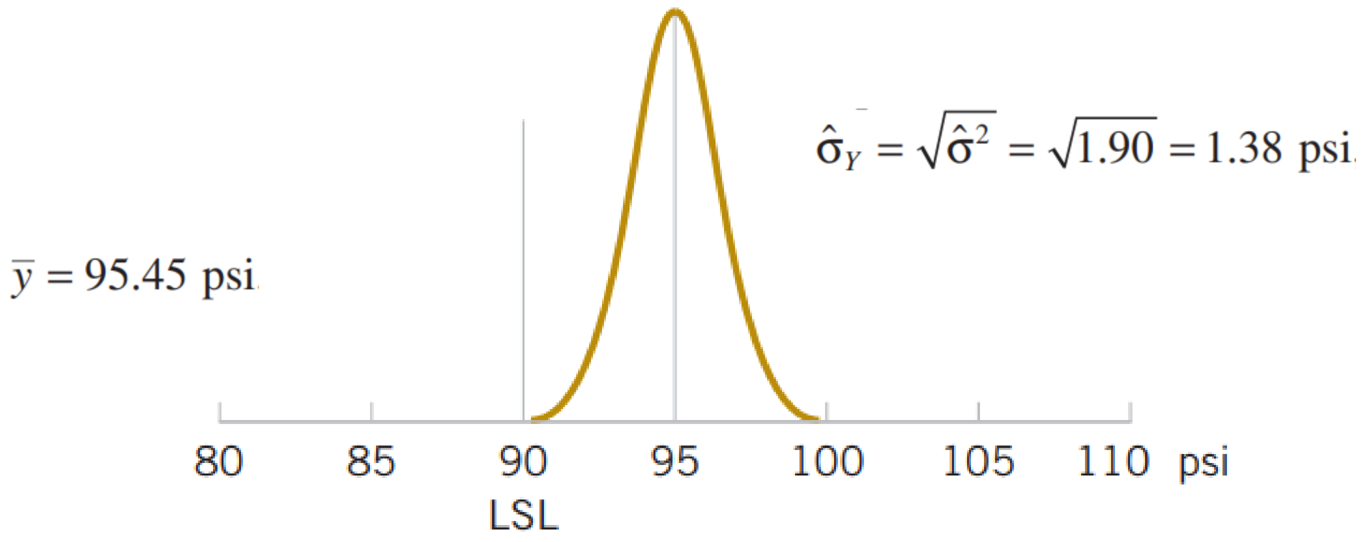
Therefore, the variance of strength in the manufacturing process is estimated by

$$\widehat{V(Y_{ij})}\ \hat{\sigma}_\tau^2 + \hat{\sigma}^2 = 6.96 + 1.90 = 8.86$$

Conclusion: Most of the variability in strength in the output product is attributable to differences between looms.



(a)



(b)

- If the lower specification limit (LSL) on strength is at 90 psi, a substantial proportion of the process output is fallout—that is, scrap or defective material that must be sold as second quality, and so on.
- This fallout is directly related to the excess variability resulting from differences between looms. Variability in loom performance could be caused by **faulty setup, poor maintenance, inadequate supervision, poorly trained operators**, and so forth.

- *Improved process, reducing the variability in strength has greatly reduced the fallout, resulting in lower cost, higher quality, a more satisfied customer, and an enhanced competitive position for the company*

13-35. ⊕ A textile mill has a large number of looms. Each loom is supposed to provide the same output of cloth per minute. To investigate this assumption, five looms are chosen at random, and their output is measured at different times. The following data are obtained:

Loom	Output (lb/min)				
1	4.0	4.1	4.2	4.0	4.1
2	3.9	3.8	3.9	4.0	4.0
3	4.1	4.2	4.1	4.0	3.9
4	3.6	3.8	4.0	3.9	3.7
5	3.8	3.6	3.9	3.8	4.0

(a) Are the looms similar in output? Use $\alpha = 0.05$.
(b) Estimate the variability between looms.
(c) Estimate the experimental error variance.
(d) Analyze the residuals from this experiment and check for model adequacy.

13-36. In the book *Bayesian Inference in Statistical Analysis* (1973, John Wiley and Sons) by Box and Tiao, the total product yield for five samples was determined randomly selected from each of six randomly chosen batches of raw material.

Batch	Yield (in grams)				
1	1545	1440	1440	1520	1580
2	1540	1555	1490	1560	1495
3	1595	1550	1605	1510	1560
4	1445	1440	1595	1465	1545
5	1595	1630	1515	1635	1625
6	1520	1455	1450	1480	1445

(a) Do the different batches of raw material significantly affect mean yield? Use $\alpha = 0.01$.
(b) Estimate the variability between batches.
(c) Estimate the variability between samples within batches.
(d) Analyze the residuals from this experiment and check for model adequacy.

Randomized Complete Block Design

- In many experimental design problems, it is necessary to design the experiment so that **the variability arising from a nuisance factor can be controlled**.

- The paired *t*-test is a procedure for comparing two treatment means when all experimental runs cannot be made under homogeneous conditions.

- The paired *t*-test is a method for reducing the background noise in the experiment by blocking out a nuisance factor effect.

- The randomized block design is an extension of the paired *t*-test to situations where the factor of interest has more than two levels; that is, more than two treatments must be compared.

Randomized Complete Block Design

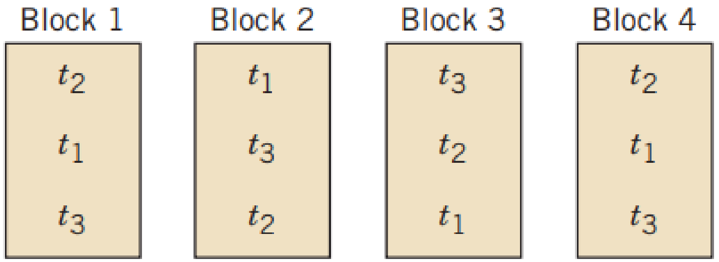


FIGURE 13-9 A randomized complete block design.

The design is called a RCBD because each block is large enough to hold all the treatments and because the actual assignment of each of the three treatments within each block is done randomly

TABLE • 13-9 A Randomized Complete Block Design

Treatments (Method)	Block (Girder)			
	1	**2**	**3**	**4**
1	y_{11}	y_{12}	y_{13}	y_{14}
2	y_{21}	y_{22}	y_{23}	y_{24}
3	y_{31}	y_{32}	y_{33}	y_{34}

The observations in this table, say, y_{ij}, represent the response obtained when method i is used on girder j.

TABLE • 13-10 A Randomized Complete Block Design with a Treatments and b Blocks

	Blocks					
Treatments	**1**	**2**	**...**	***b***	**Totals**	**Averages**
1	y_{11}	y_{12}	$\cdots$	y_{1b}	$y_{1\cdot}$	$\bar{y}_{1\cdot}$
2	y_{21}	y_{22}	$\cdots$	y_{2b}	$y_{2\cdot}$	$\bar{y}_{2\cdot}$
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$
a	y_{a1}	y_{a2}	$\cdots$	y_{ab}	$\bar{y}_{a\cdot}$	$\bar{y}_{a\cdot}$
Totals	$y_{\cdot 1}$	$y_{\cdot 2}$	$\cdots$	$y_{\cdot b}$	$\bar{y}_{\cdot\cdot}$	
Averages	$\bar{y}_{\cdot 1}$	$\bar{y}_{\cdot 2}$	$\cdots$	$\bar{y}_{\cdot b}$		$\bar{y}_{\cdot\cdot}$

The general procedure for a **RCBD** consists of selecting b blocks and running a complete replicate of the experiment in each block. The data that result from running a RCBD for investigating a single factor with a levels and b blocks are shown in Table 13-10. There are a observations (one per factor level) in each block, and the order in which these observations are run is randomly assigned within the block.

We now describe the statistical analysis for the RCBD. Suppose that a single factor with a levels is of interest and that the experiment is run in b blocks. The observations may be represented by the **linear statistical model**

$$Y_{ij} = \mu + \tau_i + \beta_j + \epsilon_{ij} \begin{cases} i = 1, 2, \ldots, a \\ j = 1, 2, \ldots, b \end{cases} \tag{13-26}$$

where μ is an overall mean, τ_i is the effect of the ith treatment, β_j is the effect of the jth block, and ϵ_{ij} is the random error term, which is assumed to be normally and independently distributed with mean zero and variance σ^2. Furthermore, the treatment and block effects are defined as deviations from the overall mean, so $\sum_{j=1}^{a} \tau_i = 0$ and $\sum_{j=1}^{b} \beta_j = 0$. This was the same type of definition used for completely randomized experiments in Section 13-2. We also assume that treatments and blocks do not interact. That is, the effect of treatment i is the same regardless of which block (or blocks) in which it is tested. We are interested in testing the equality of the treatment effects.

$$H_0: \tau_1 = \tau_2 = \cdots = \tau_a = 0 \qquad H_1: \tau_i \neq 0 \text{ at least one } i$$

The analysis of variance can be extended to the RCBD. The procedure uses a sum of squares identity that partitions the total sum of squares into three components.

ANOVA Sums of Squares Identity: Randomized Complete Block Experiment

The **sum of squares identity for the randomized complete block design** is

$$\sum_{i=1}^{a}\sum_{j=1}^{b}(y_{ij}-\bar{y}_{..})^2 = b\sum_{i=1}^{a}(\bar{y}_{i.}-\bar{y}_{..})^2 + a\sum_{j=1}^{b}(\bar{y}_{.j}-\bar{y}_{..})^2 + \sum_{i=1}^{a}\sum_{j=1}^{b}(y_{ij}-\bar{y}_{.j}-\bar{y}_{i.}+\bar{y}_{..})^2 \qquad (13\text{-}27)$$

or symbolically

$$SS_T = SS_{\text{Treatments}} + SS_{\text{Blocks}} + SS_E$$

Expected Mean Squares: Randomized Complete Block Experiment

$$E(MS_{\text{Treatments}}) = \sigma^2 + \frac{b\sum_{i=1}^{a}\tau_i^2}{a-1}$$

$$E(MS_{\text{Blocks}}) = \sigma^2 + \frac{a\sum_{j=1}^{b}\beta_j^2}{b-1}$$

$$E(MS_E) = \sigma^2$$

Therefore, if the null hypothesis H_0 is true so that all treatment effects $\tau_i = 0$, $MS_{\text{Treatments}}$ is an unbiased estimator of σ^2, and if H_0 is false, $MS_{\text{Treatments}}$ overestimates σ^2. The mean square for error is always an unbiased estimate of σ^2. To test the null hypothesis that the treatment effects are all zero, we use the ratio

$$F_0 = \frac{MS_{\text{Treatments}}}{MS_E} \tag{13-28}$$

which has an F-distribution with $a-1$ and $(a-1)(b-1)$ degrees of freedom if the null hypothesis is true. We would reject the null hypothesis at the a-level of significance if the computed value of the test statistic in Equation 13-28 is $f_0 > f_{\alpha,a-1,(a-1)(b-1)}$.

Computing Formulas for ANOVA: Randomized Block Experiment

The computing formulas for the sums of squares in the analysis of variance for a RCBD are

$$SS_T = \sum_{i=1}^{a}\sum_{j=1}^{b} y_{ij}^2 - \frac{y_{..}^2}{ab} \tag{13-29}$$

$$SS_{\text{Treatments}} = \frac{1}{b}\sum_{i=1}^{a} y_{i.}^2 - \frac{y_{..}^2}{ab} \tag{13-30}$$

$$SS_{\text{Blocks}} = \frac{1}{a}\sum_{j=1}^{b} y_{.j}^2 - \frac{y_{..}^2}{ab} \tag{13-31}$$

and

$$SS_E = SS_T - SS_{\text{Treatments}} - SS_{\text{Blocks}} \tag{13-32}$$

TABLE • 13-11 ANOVA for a Randomized Complete Block Design

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0
Treatments	$SS_{\text{Treatments}}$	$a-1$	$\frac{SS_{\text{Treatments}}}{a-1}$	$\frac{MS_{\text{Treatments}}}{MS_E}$
Blocks	SS_{Blocks}	$b-1$	$\frac{SS_{\text{Blocks}}}{b-1}$	
Error	SS_E (by subtraction)	$(a-1)(b-1)$	$\frac{SS_E}{(a-1)(b-1)}$	
Total	SS_T	$ab-1$		

Example 13-5 **Fabric Strength** An experiment was performed to determine the effect of four different chemicals on the strength of a fabric. These chemicals are used as part of the permanent press finishing process. Five fabric samples were selected, and a RCBD was run by testing each chemical type once in random order on each fabric sample. The data are shown in Table 13-12. We test for differences in means using an ANOVA with $\alpha = 0.01$.

TABLE • 13-12 Fabric Strength Data—Randomized Complete Block Design

Chemical Type	Fabric Sample 1	2	3	4	5	Treatment Totals $y_{i\cdot}$	Treatment Averages $\bar{y}_{i\cdot}$
1	1.3	1.6	0.5	1.2	1.1	5.7	1.14
2	2.2	2.4	0.4	2.0	1.8	8.8	1.76
3	1.8	1.7	0.6	1.5	1.3	6.9	1.38
4	3.9	4.4	2.0	4.1	3.4	17.8	3.56
Block totals $y_{\cdot j}$	9.2	10.1	3.5	8.8	7.6	39.2($y_{\cdot\cdot}$)	
Block averages $\bar{y}_{\cdot j}$	2.30	2.53	0.88	2.20	1.90		1.96($\bar{y}_{\cdot\cdot}$)

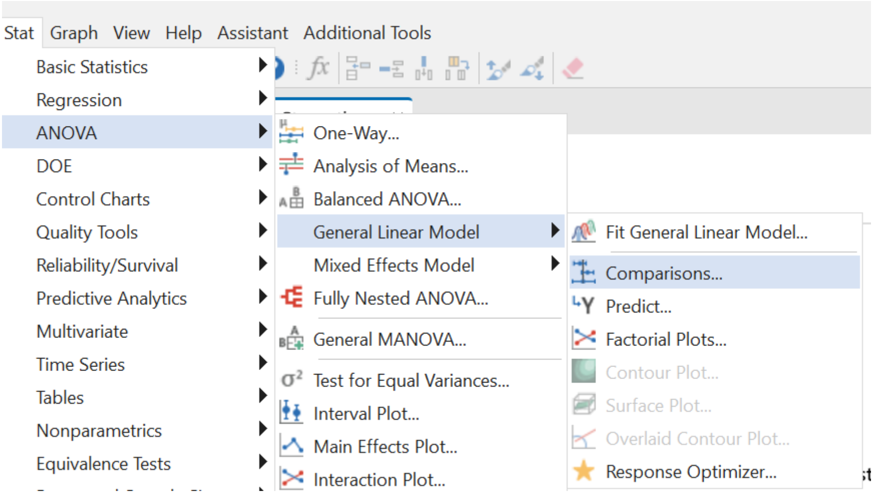
$$SS_T = \sum_{i=1}^{4}\sum_{j=1}^{5} y_{ij}^2 - \frac{y_{..}^2}{ab} = (1.3)^2 + (1.6)^2 + \cdots + (3.4)^2 - \frac{(39.2)^2}{20} = 25.69$$

$$SS_{\text{Treatments}} = \sum_{i=1}^{4} \frac{y_{i.}^2}{b} - \frac{y_{..}^2}{ab} = \frac{(5.7)^2 + (8.8)^2 + (6.9)^2 + (17.8)^2}{5} - \frac{(39.2)^2}{20} = 18.04$$

$$SS_{\text{Blocks}} = \sum_{j=1}^{5} \frac{y_{.j}^2}{a} - \frac{y_{..}^2}{ab} = \frac{(9.2)^2 + (10.1)^2 + (3.5)^2 + (8.8)^2 + (7.6)^2}{4} - \frac{(39.2)^2}{20} = 6.69$$

$$SS_E = SS_T - SS_{\text{Blocks}} - SS_{\text{Treatments}} = 25.69 - 6.69 - 18.04 = 0.96$$

$$f_0 = 75.13 > f_{0.01,3,12} = 5.95$$



we conclude that there is a significant difference in the chemical types so far as their effect on strength is concerned.

Table 13-14 Computer Output for the Randomized Complete Block Design in Example 13-5

Factor	Type	Levels	Values				
Chemical	fixed	4	1	2	3	4	
Fabric S	fixed	5	1	2	3	4	5

Analysis of Variance for Strength

Source	DF	SS	MS	F	P
Chemical	3	18.0440	6.0147	75.89	0.000
Fabric	4	6.6930	1.6733	21.11	0.000
Error	12	0.9510	0.0792		
Total	19	25.6880			

F-test with denominator: Error
Denominator MS = 0.079250 with 12 degrees of freedom

Numerator	DF	MS	F	P
Chemical	3	6.015	75.89	0.000
Fabric S	4	1.673	21.11	0.000

We illustrate Fisher's LSD method. The four chemical type averages from Example 13-5 are:

$$\bar{y}_{1\cdot} = 1.14 \qquad \bar{y}_{2\cdot} = 1.76 \qquad \bar{y}_{3\cdot} = 1.38 \qquad \bar{y}_{4\cdot} = 3.56$$

Each treatment average uses $b = 5$ observations (one from each block). We use $\alpha = 0.05$, so $t_{0.025,12} = 2.179$. Therefore, the value of the LSD is

$$\text{LSD} = t_{0.025,12}\sqrt{\frac{2MS_E}{b}} = 2.179\sqrt{\frac{2(0.08)}{5}} = 0.39$$

Any pair of treatment averages that differ by 0.39 or more indicates that this pair of treatment means is significantly different. The comparisons follow:

$$4 \text{ vs. } 1 = \bar{y}_{4\cdot} - \bar{y}_{1\cdot} = 3.56 - 1.14 = 2.42 > 0.39$$
$$4 \text{ vs. } 3 = \bar{y}_{4\cdot} - \bar{y}_{3\cdot} = 3.56 - 1.38 = 2.18 > 0.39$$
$$4 \text{ vs. } 2 = \bar{y}_{4\cdot} - \bar{y}_{2\cdot} = 3.56 - 1.76 = 1.80 > 0.39$$
$$2 \text{ vs. } 1 = \bar{y}_{2\cdot} - \bar{y}_{1\cdot} = 1.76 - 1.14 = 0.62 > 0.39$$
$$2 \text{ vs. } 3 = \bar{y}_{2\cdot} - \bar{y}_{3\cdot} = 1.76 - 1.38 = 0.38 < 0.39$$
$$3 \text{ vs. } 1 = \bar{y}_{3\cdot} - \bar{y}_{1\cdot} = 1.38 - 1.14 = 0.24 < 0.39$$

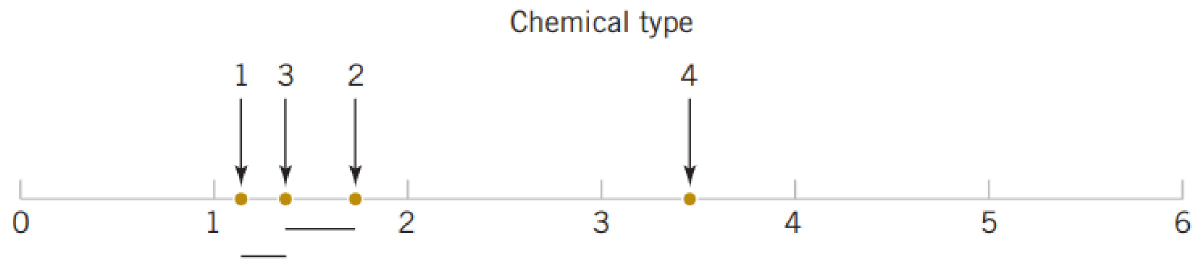


FIGURE 13-10 Results of Fisher's LSD method.

RESIDUAL ANALYSIS AND MODEL CHECKING

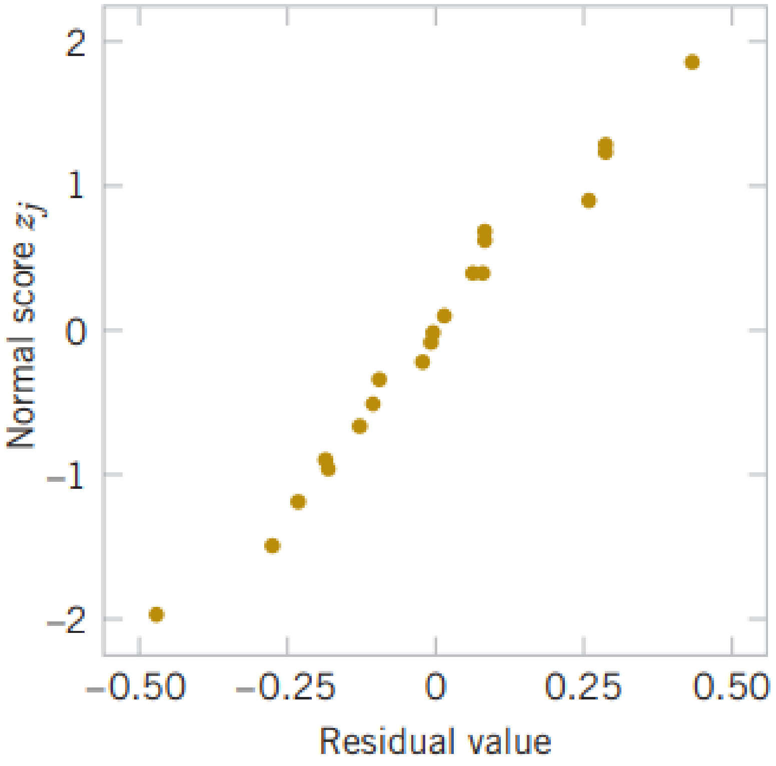


FIGURE 13-11 Normal probability plot of residuals from the randomized complete block design.

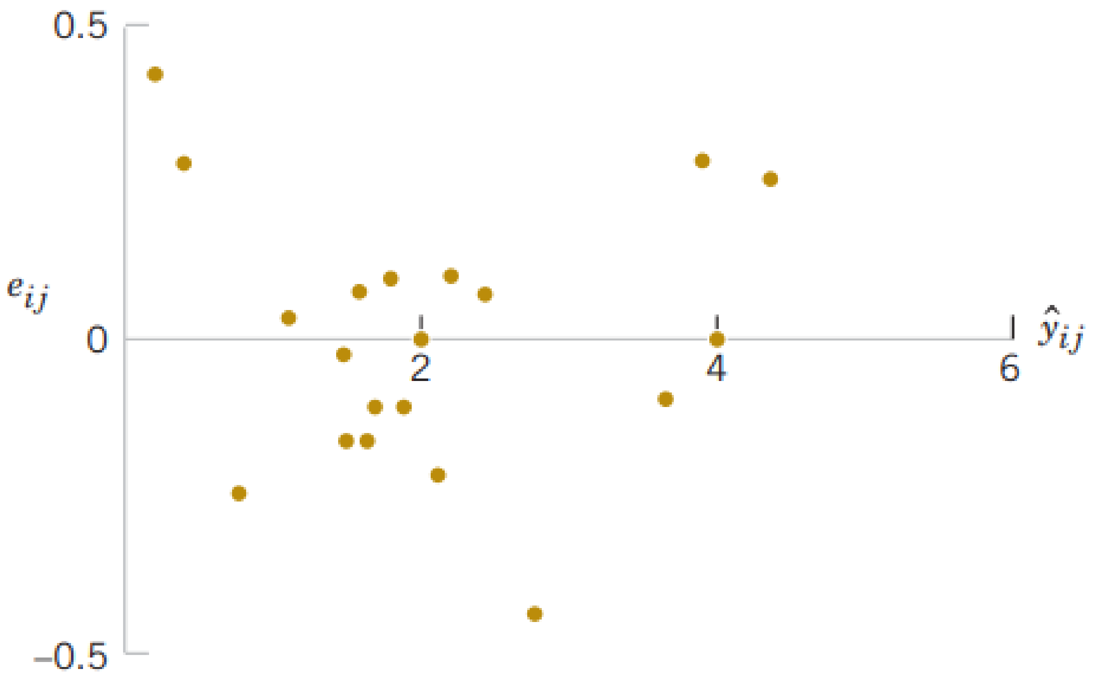
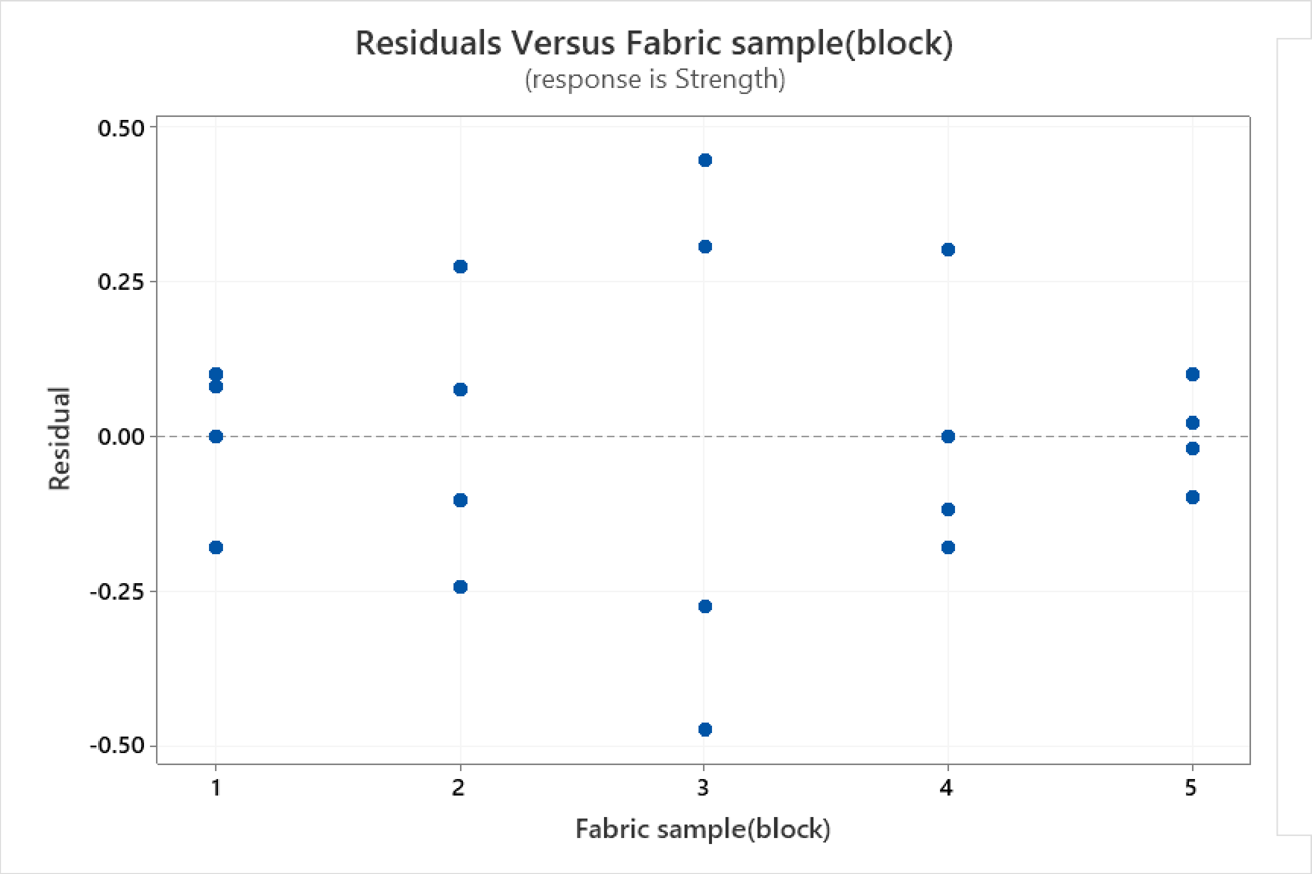
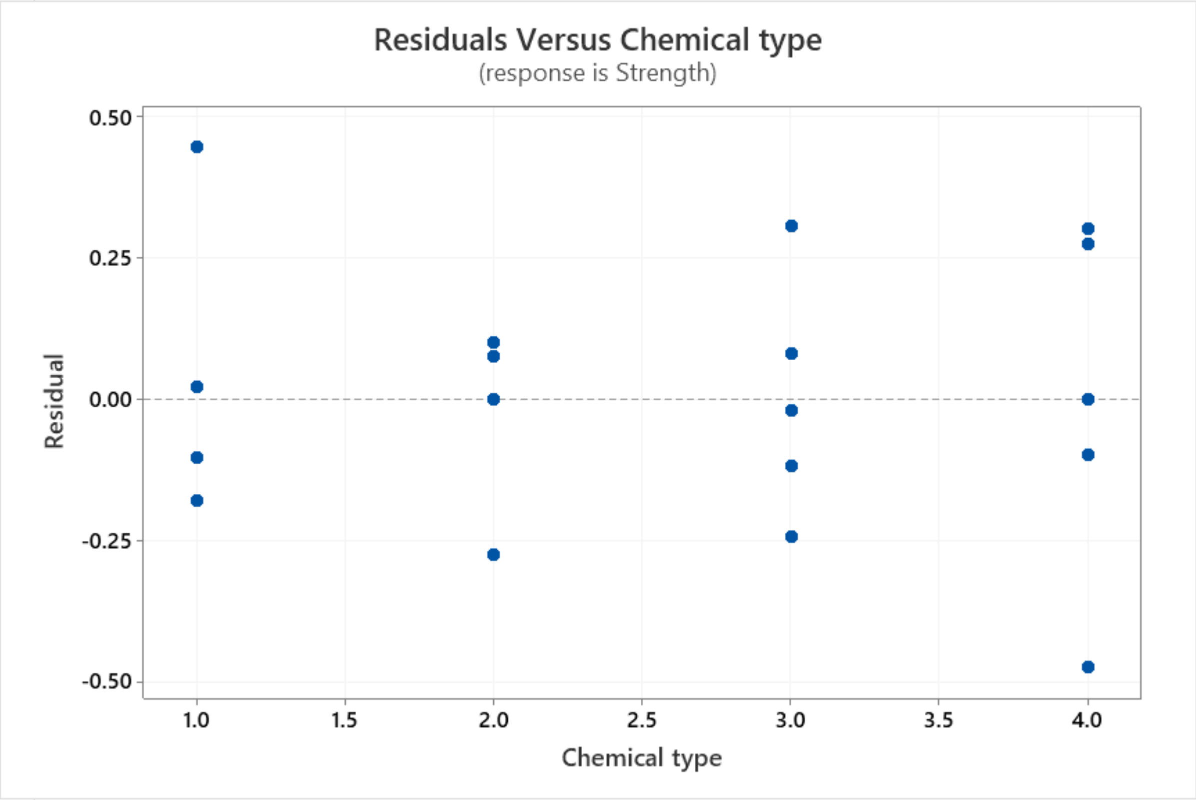


FIGURE 13-14 Residuals versus $\hat{y}_{ij}$ from the randomized complete block design.





When treated with the four chemicals, there is some indication that fabric sample (block) 3 has greater variability in strength than the other samples

Chemical type 4, which provides the greatest strength, also has somewhat more variability in strength

Follow-up experiments may be necessary to confirm these findings if they are potentially important

13-42. Consider the following computer output from a RCBD.

Source	DF	SS	MS	F	P
Factor	?	193.800	64.600	?	?
Block	3	464.218	154.739		
Error	?	?	4.464		
Total	15	698.190			

(a) How many levels of the factor were used in this experiment?
(b) How many blocks were used in this experiment?
(c) Fill in the missing information. Use bounds for the *P*-value.
(d) What conclusions would you draw if $\alpha = 0.05$? What would you conclude if $\alpha = 0.01$?

13-48. In *Design and Analysis of Experiments*, 8th edition (John Wiley & Sons, 2012), D. C. Montgomery described an experiment that determined the effect of four different types of tips in a hardness tester on the observed hardness of a metal alloy. Four specimens of the alloy were obtained, and each tip was tested once on each specimen, producing the following data:

	Specimen			
Type of Tip	**1**	**2**	**3**	**4**
1	9.3	9.4	9.6	10.0
2	9.4	9.3	9.8	9.9
3	9.2	9.4	9.5	9.7
4	9.7	9.6	10.0	10.2

(a) Is there any difference in hardness measurements between the tips?

(b) Use Fisher's LSD method to investigate specific differences between the tips.

(c) Analyze the residuals from this experiment.

14

Design of Experiments with Several Factors

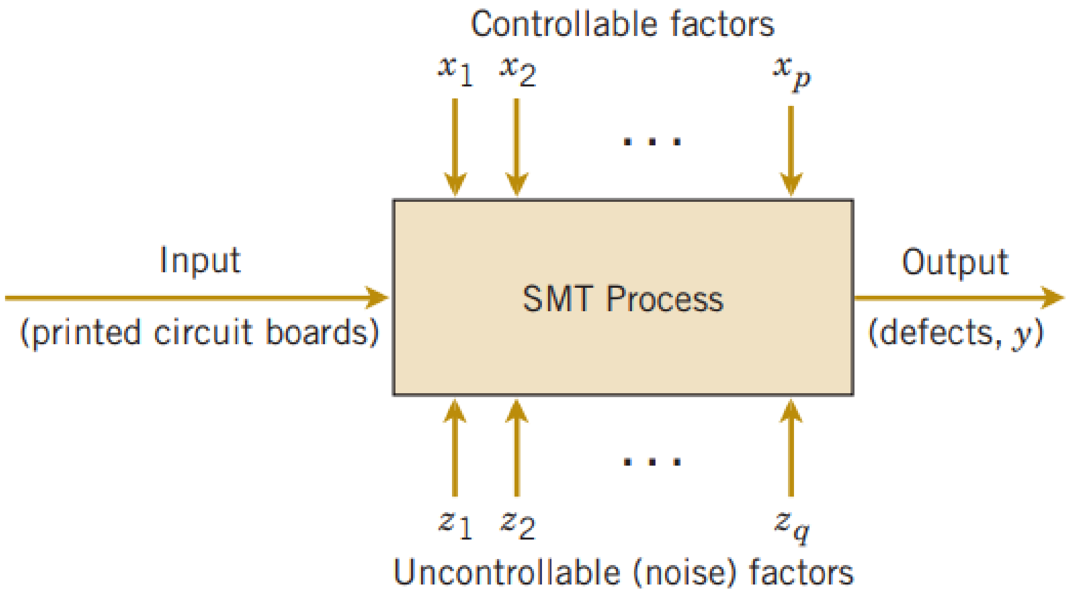
Chapter Learning Outcomes

- Design and conduct engineering experiments involving several factors using the factorial design approach
- Analyze and interpret main effects and interactions
- Understand how to use the ANOVA to analyze the data from these experiments
- Assess model adequacy with residual plots
- Use the two-level series of factorial designs
- Understand how to run two-level factorial design in blocks
- Design and conduct two-level fractional factorial designs
- Use center points to test for curvature in two-level factorial designs
- Use response surface methodology for process optimization experiments

- We will focus on experiments that include two or more factors that the experimenter thinks may be important.

- A factorial experiment is a powerful technique for this type of problem. Generally, in a factorial experimental design, experimental trials (or runs) are performed at all combinations of factor levels.

- For example, if a chemical engineer is interested in investigating the effects of reaction time and reaction temperature on the yield of a process, and if two levels of time (1.0 and 1.5 hours) and two levels of temperature (125 and 150°F) are considered important, a factorial experiment would consist of making experimental runs at each of the four possible combinations of these levels of reaction time and reaction temperature.



The uncontrollable factors noise factors

Optimization Experiment

In a characterization experiment, we are interested in determining which factors affect the response. A logical next step is to determine the region in the important factors that leads to an optimum response. For example, if the response is cost, we look for a region of minimum cost. This leads to an optimization experiment.

Factorial Experiment

- When several factors are of interest in an experiment, a factorial experiment should be used.

Factorial Experiment

> By **factorial experiment**, we mean that in each complete trial or replicate of the experiment, all possible combinations of the levels of the factors are investigated.

- Thus, if there are two factors A and B with ***a*** levels of factor A and ***b*** levels of factor B, each replicate contains all ***ab*** treatment combinations.

- The effect of a factor is defined as the change in response produced by a change in the level of the factor. It is called a **main effect** because it refers to the primary factors in the study.

- Factorial experiment with two factors, A and B, each at two levels (A_{low}, A_{high}, B_{low}, B_{high})

- The main effect of factor A is the difference between the average response at the high level of A and the average response at the low level of A.

TABLE • A Factorial Experiment with Two Factors

Factor A	Factor B: B_{low}	Factor B: B_{high}
A_{low}	10	20
A_{high}	30	40

$$A = \frac{30+40}{2} - \frac{10+20}{2} = 20$$

$$B = \frac{20+40}{2} - \frac{10+30}{2} = 10$$

Factor A	Factor B	
	B_{low}	B_{high}
A_{low}	10	20
A_{high}	30	0

- When the difference in response between the levels of one factor *is not the same* at all levels of the other factors. When this occurs, there is an interaction between the factors

- At the low level of factor B, the A effect is $A = 30 - 10 = 20$

- At the high level of factor B, the A effect is $A = 0 - 20 = -20$

Because the effect of A depends on the level chosen for factor B, there is interaction between A and B.

- When an interaction is large, the corresponding main effects have very little practical meaning. For example, the main effect of A as

$$A = \frac{30 + 0}{2} - \frac{10 + 20}{2} = 0$$

We conclude that there is no factor A effect.

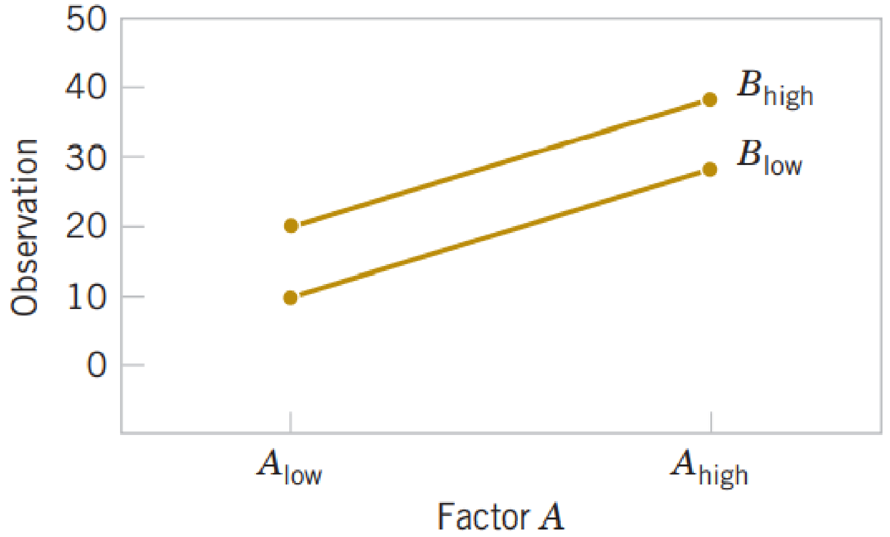
However, when we examined the effects of A at different levels of factor B, we saw that this was not the case. The effect of factor A depends on the levels of factor B. Thus, knowledge of the AB interaction is more useful than knowledge of the main effect. A significant interaction can mask the significance of main effects.

	Factor *B*	
Factor *A*	B_{low}	B_{high}
A_{low}	10	20
A_{high}	30	0

Two-factor interaction plots.

Factor A	Factor B: B_{low}	Factor B: B_{high}
A_{low}	10	20
A_{high}	30	40

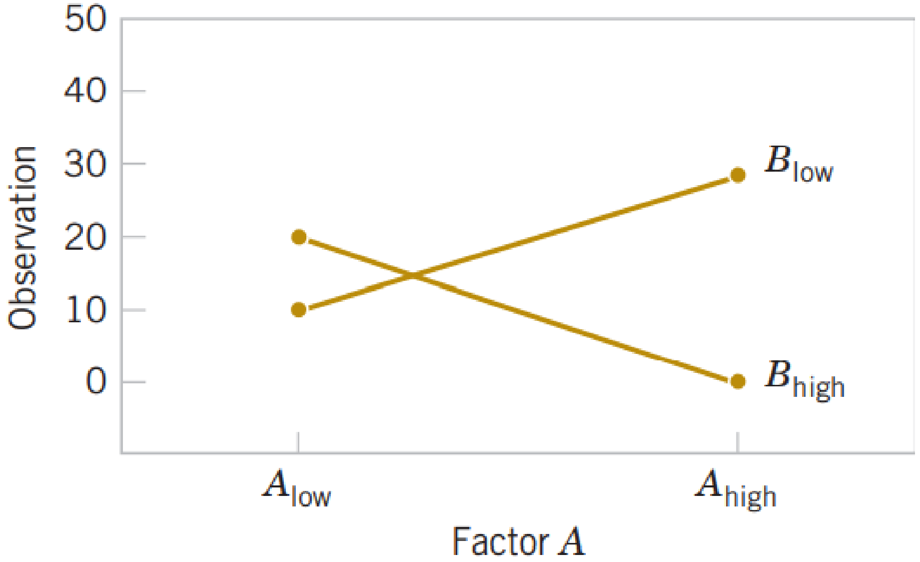
$$AB = \frac{20+30}{2} - \frac{10+40}{2} = 0$$



Factorial experiment, no interaction.

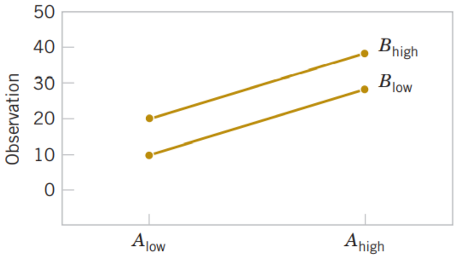
Factor A	Factor B: B_{low}	Factor B: B_{high}
A_{low}	10	20
A_{high}	30	0

$$AB = \frac{20+30}{2} - \frac{10+0}{2} = 20$$

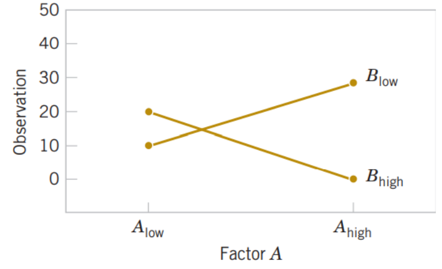


Factorial experiment, with interaction.

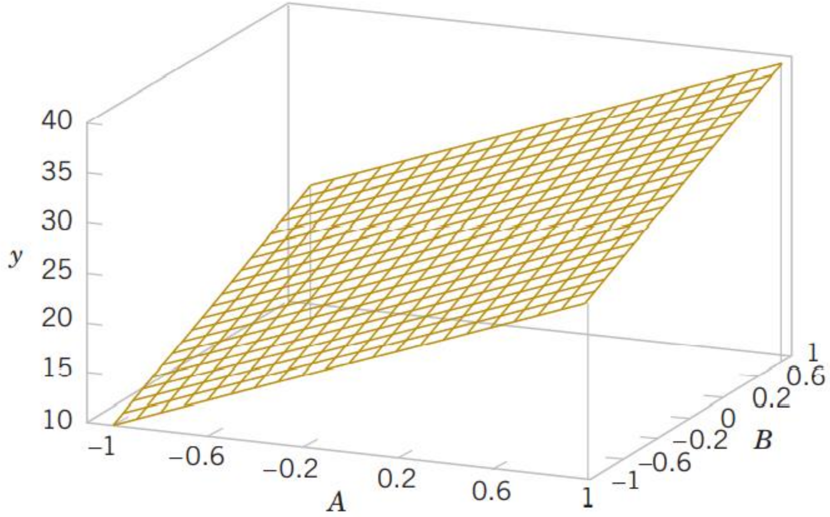
Three-dimensional surface plot

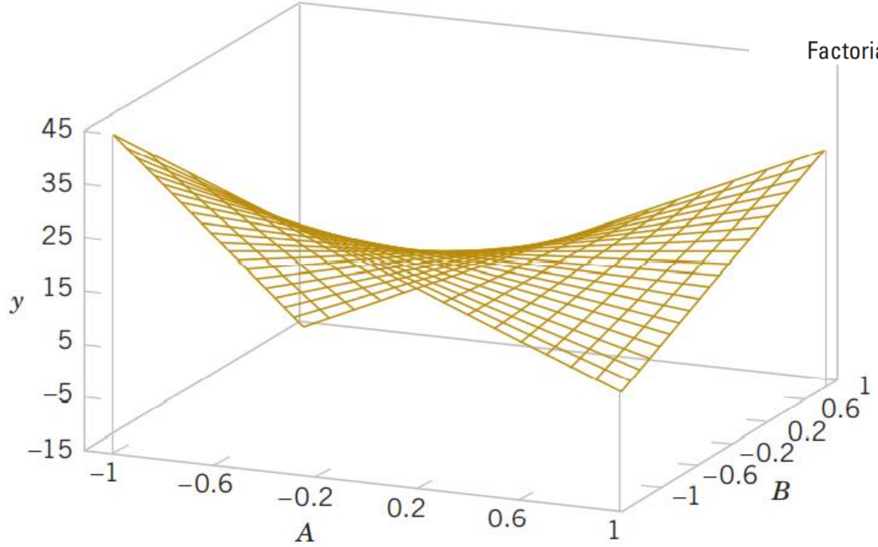


Factorial experim



Factorial experiment, with interaction.

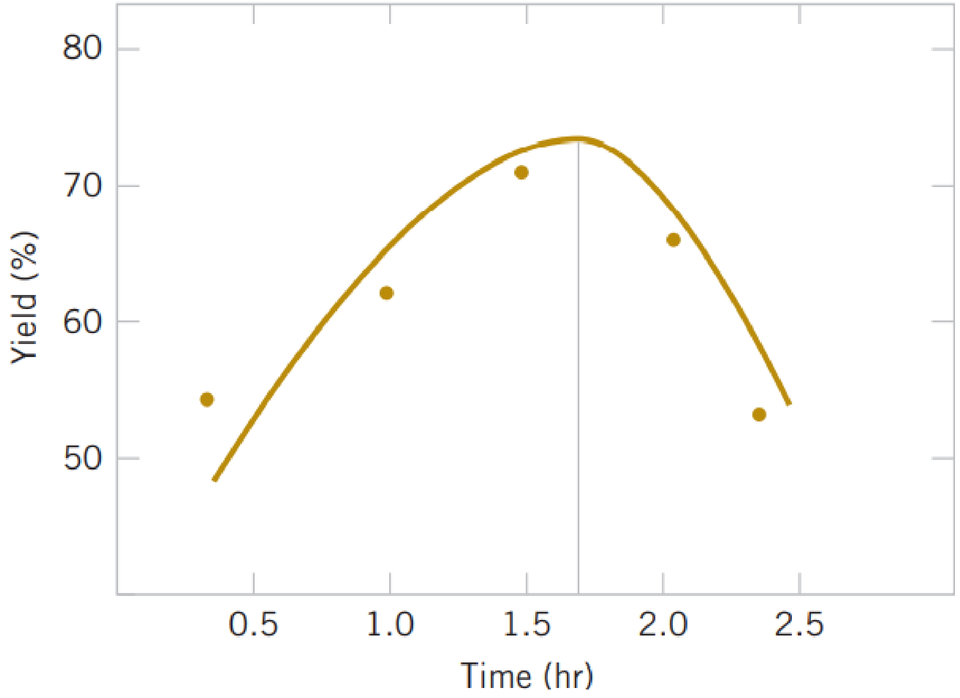




- ❑ The slope of the plane in the A and B directions is proportional to the main effects of factors A and B, respectively
- ❑ The effect of interaction in these data is to "twist" the plane so that there is curvature in the response function.

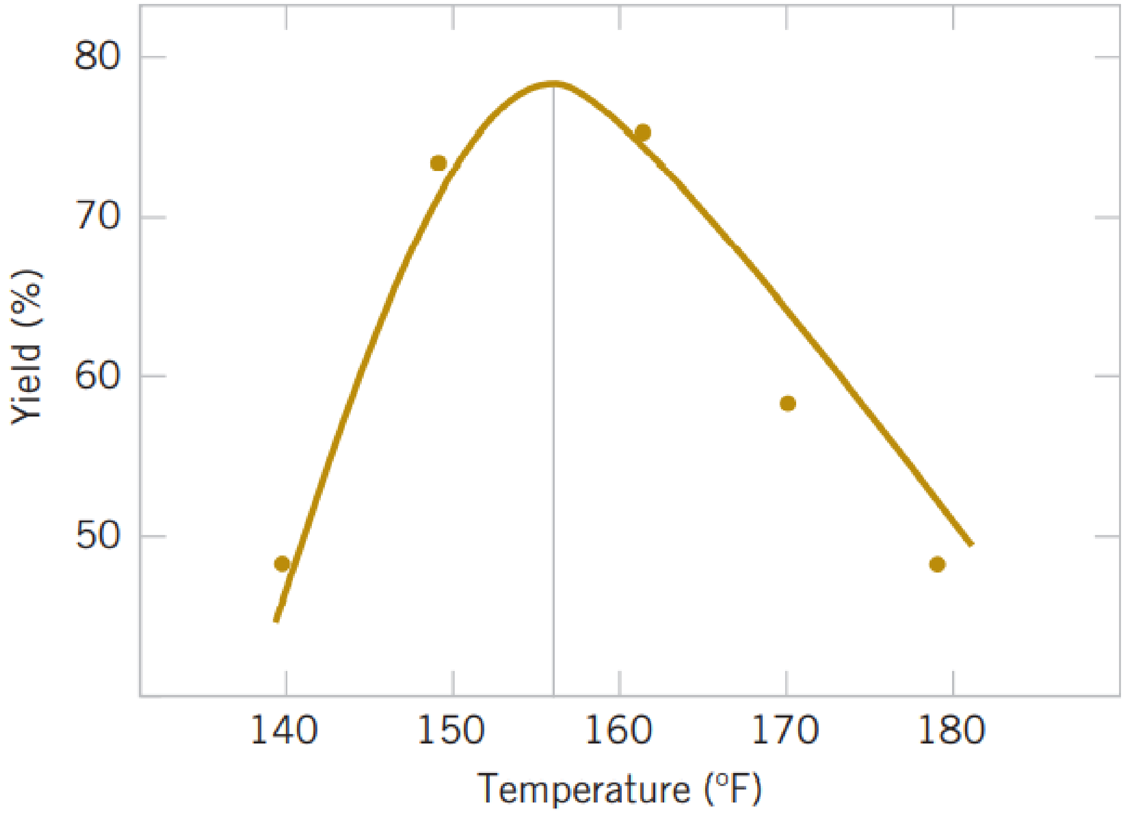
***Factorial experiments are the only way to discover interactions between variables*.**

An alternative to the factorial design that is (unfortunately) used in practice is to change the factors ***one at a time*** rather than to vary them simultaneously. To illustrate this one-factor-at-a-time procedure, suppose that an engineer is interested in finding the values of temperature and pressure that maximize yield in a chemical process. Suppose that we fix temperature at 155°F (the current operating level) and perform five runs at different levels of time, say, 0.5, 1.0, 1.5, 2.0, and 2.5 hours.



This figure indicates that maximum yield is achieved at about 1.7 hours of reaction time.

To optimize temperature, the engineer then fixes time at 1.7 hours (the apparent optimum) and performs five runs at different temperatures, say, 140, 150, 160, 170, and 180°F



Maximum yield occurs at about 155°F. Therefore, we would conclude that running the process at 155°F and 1.7 hours is the best set of operating conditions, resulting in yields of around 75%.

The failure to discover the importance of the shorter reaction times is particularly important because this could have significant impact on production volume or capacity, production planning, manufacturing cost, and total productivity.

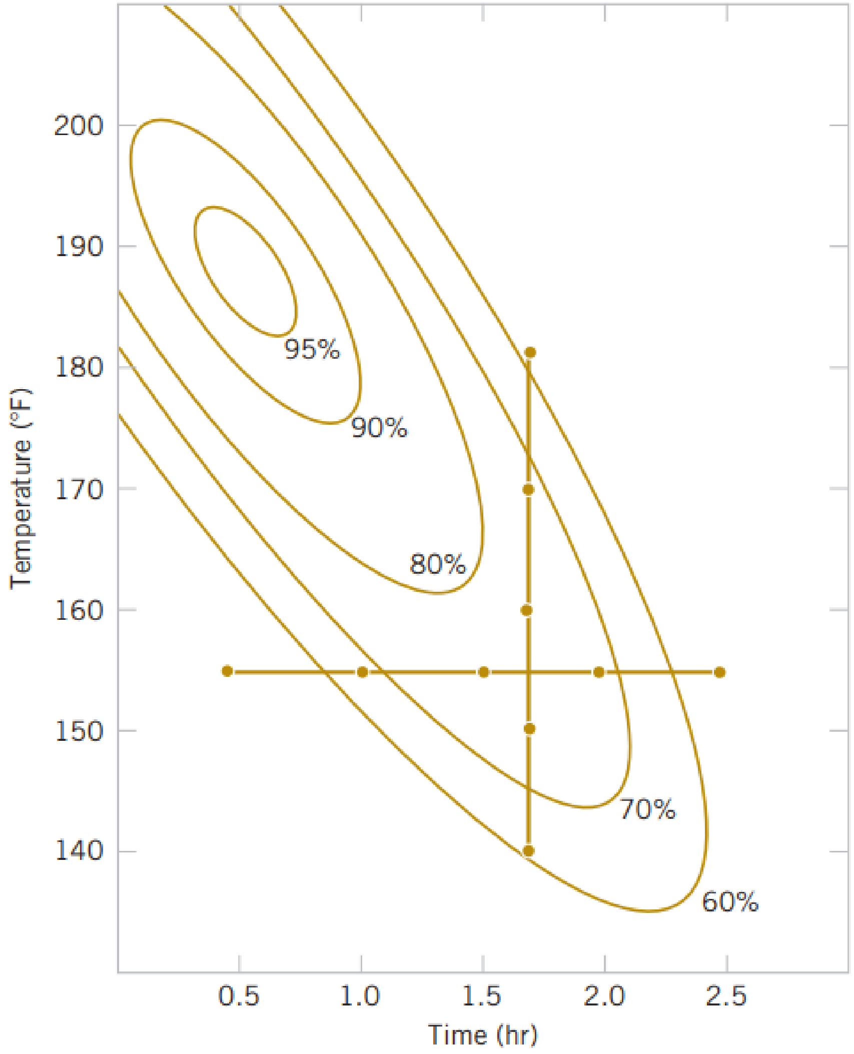


FIGURE 14-9 Optimization experiment using the one-factor-at-a-time method.

The one-factor-at-a-time approach **has failed** here because it cannot detect the interaction between temperature and time. Factorial experiments are the only way to detect interactions. Furthermore, the one-factor-at-a-time method **is inefficient**. It requires more experimentation than a factorial, and as we have just seen, there is no assurance that it will produce the correct results.

Two-Factor Factorial Experiments

Two-Factor Factorial Experiments

- There are ***a*** levels of factor A and ***b*** levels of factor B.
- The experiment has *n* replicates, and each replicate contains all ***ab*** treatment combinations.
- The observation in the *ij*th cell for the *k*th replicate is denoted by y_{ijk}.

$$Y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \epsilon_{ijk} \begin{cases} i = 1, 2, \ldots, a \\ j = 1, 2, \ldots, b \\ k = 1, 2, \ldots, n \end{cases}$$

where μ is the overall mean effect

τ_i is the effect of the ith level of factor A

β_j is the effect of the jth level of factor B

$(\tau\beta)_{ij}$ is the effect of the interaction between A and B

ϵ_{ijk} is a random error component having a normal distribution with mean 0 and variance σ^2

Statistical analysis of the fixed-effects model

TABLE Data Arrangement for a Two-Factor Factorial Design

		Factor *B*					
		1	**2**	⋯	***b***	**Totals**	**Averages**
	1	$y_{111}, y_{112}, \cdots, y_{11n}$	$y_{121}, y_{122}, \cdots, y_{12n}$		$y_{1b1}, y_{1b2}, \cdots, y_{1bn}$	$\bar{y}_{1\cdot\cdot}$	$\bar{y}_{1\cdot\cdot}$
Factor *A*	2	$y_{211}, y_{212}, \cdots, y_{21n}$	$y_{221}, y_{222}, \cdots, y_{22n}$		$y_{2b1}, y_{2b2}, \cdots, y_{2bn}$	$\bar{y}_{2\cdot\cdot}$	$\bar{y}_{2\cdot\cdot}$
	⋮						
	a	$y_{a11}, y_{a12}, \cdots, y_{a1n}$	$y_{a21}, y_{a22}, \cdots, y_{a2n}$		$y_{ab1}, y_{ab2}, \cdots, y_{abn}$	$y_{a\cdot\cdot}$	$\bar{y}_{a\cdot\cdot}$
Totals		$y_{\cdot 1\cdot}$	$y_{\cdot 2\cdot}$		$y_{\cdot b\cdot}$	$y_{\cdot\cdot\cdot}$	
Averages		$\bar{y}_{\cdot 1\cdot}$	$\bar{y}_{\cdot 2\cdot}$		$\bar{y}_{\cdot b\cdot}$		$\bar{y}_{\cdot\cdot\cdot}$

Notation for Totals and Means

$$y_{i\cdot\cdot} = \sum_{j=1}^{b}\sum_{k=1}^{n} y_{ijk} \qquad \bar{y}_{i\cdot\cdot} = \frac{y_{i\cdot\cdot}}{bn} \qquad i = 1, 2, \ldots, a$$

$$y_{j\cdot\cdot} = \sum_{i=1}^{a}\sum_{k=1}^{n} y_{ijk} \qquad \bar{y}_{\cdot j\cdot} = \frac{y_{\cdot j\cdot}}{an} \qquad j = 1, 2, \ldots, b$$

$$y_{ij\cdot\cdot\cdot} = \sum_{k=1}^{n} y_{ijk} \qquad \bar{y}_{ij\cdot} = \frac{y_{ij\cdot}}{n} \qquad i = 1, 2, \ldots, a, \; j = 1, 2, \ldots, b$$

$$y_{\cdot\cdot\cdot} = \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n} y_{ijk} \qquad \bar{y}_{\cdot\cdot\cdot} = \frac{y_{\cdot\cdot\cdot}}{abn} \qquad i = 1, 2, \ldots, a$$

The hypotheses that we will test are as follows:

$$H_0: \tau_1 = \tau_2 = \ldots = \tau_a = 0 \quad \text{(no main effect of factor } A\text{)}$$
$$H_1: \text{at least one } \tau_i \neq 0$$
$$H_0: \beta_1 = \beta_2 = \ldots = \beta_b = 0 \quad \text{(no main effect of factor } B\text{)}$$
$$H_1: \text{at least one } \beta_j \neq 0$$
$$H_0: (\tau\beta)_{11} = (\tau\beta)_{12} = \ldots = (\tau\beta)_{ab} = 0 \quad \text{(no interaction)}$$
$$H_1: \text{at least one } (\tau\beta)_{ij} \neq 0$$

$$SS_T = \sum_{i=1}^{a} \sum_{j=1}^{b} \sum_{k=1}^{n} (y_{ijk} - \bar{y}_{\ldots})^2$$

ANOVA Sum of Squares Identity: Two Factors

The **sum of squares identity for a two-factor ANOVA is**

$$\sum_{i=1}^{a} \sum_{j=1}^{b} \sum_{k=1}^{n} (y_{ijk} - \bar{y}_{\ldots})^2 = bn \sum_{i=1}^{a} (\bar{y}_{i\cdot\cdot} - \bar{y}_{\ldots})^2 + an \sum_{j=1}^{b} (\bar{y}_{\cdot j \cdot} - \bar{y}_{\ldots})^2$$
$$+ n \sum_{i=1}^{a} \sum_{j=1}^{b} (\bar{y}_{ij\cdot} - \bar{y}_{i\cdot\cdot} - \bar{y}_{\cdot j\cdot} + \bar{y}_{\ldots})^2 + \sum_{i=1}^{a} \sum_{j=1}^{b} \sum_{k=1}^{n} (y_{ijk} - \bar{y}_{ij\cdot})^2$$

or symbolically,

$$SS_T = SS_A + SS_B + SS_{AB} + SS_E$$

$$MS_A = \frac{SS_A}{a-1} \qquad MS_B = \frac{SS_B}{b-1} \qquad MS_{AB} = \frac{SS_{AB}}{(a-1)(b-1)} \qquad MS_E = \frac{SS_E}{ab(n-1)}$$

Expected Values of Mean Squares: Two Factors

$$E(MS_A) = E\left(\frac{SS_A}{a-1}\right) = \sigma^2 + \frac{bn\sum_{i=1}^{a}\tau_i^2}{a-1} \qquad E(MS_B) = E\left(\frac{SS_B}{b-1}\right) = \sigma^2 + \frac{an\sum_{j=1}^{a}\beta_j^2}{b-1}$$

$$E(MS_{AB}) = E\left(\frac{SS_{AB}}{(a-1)(b-1)}\right) = \sigma^2 + \frac{n\sum_{i=1}^{a}\sum_{j=1}^{b}(\tau\beta)_{ij}^2}{(a-1)(b-1)}$$

$$E(MS_E) = E\left(\frac{SS_E}{ab(n-1)}\right) = \sigma^2$$

To test that the row factor effects are all equal to zero (H_0: $\tau_i = 0$), we would use the ratio

F Test for Factor A

$$F_0 = \frac{MS_A}{MS_E}$$

which has an F distribution with $a - 1$ and $ab(n - 1)$ degrees of freedom if H_0: $\tau_i = 0$ is true. This null hypothesis is rejected at the α level of significance if $f_0 > f_{\alpha, a-1, ab(n-1)}$.

to test the hypothesis that all the column factor effects are equal to 0 (H_0: $\beta_j = 0$), we would use the ratio

F Test for Factor B

$$F_0 = \frac{MS_B}{MS_E}$$

which has an F distribution with $b - 1$ and $ab(n - 1)$ degrees of freedom if H_0: $\beta_j = 0$ is true. This null hypothesis is rejected at the α level of significance if $f_0 > f_{\alpha, b-1, ab(n-1)}$.

test the hypothesis $H_0: (\tau\beta)_{ij} = 0$, which is the hypothesis that all interaction effects are 0, we use the ratio

F Test for AB Interaction

$$F_0 = \frac{MS_{AB}}{MS_E}$$

which has an F distribution with $(a-1)(b-1)$ and $ab(n-1)$ degrees of freedom if the null hypothesis $H_0: (\tau\beta)_{ij} = 0$. This hypothesis is rejected at the a level of significance if $f_0 > f_{\alpha,(a-1)(b-1),ab(n-1)}$.

It is usually best to conduct the test for interaction first and then to evaluate the main effects. If interaction is not significant, interpretation of the tests on the main effects is straightforward. However, when interaction is significant, the main effects of the factors involved in the interaction may not have much practical interpretative value. Knowledge of the interaction is usually more important than knowledge about the main effects.

Computing Formulas for ANOVA: Two Factors

Computing formulas for the sums of squares in a two-factor analysis of variance:

$$SS_T = \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n} y^2_{ijk} - \frac{y^2_{\cdots}}{abn}$$

$$SS_A = \sum_{i=1}^{a}\frac{y^2_{i\cdot\cdot}}{bn} - \frac{y^2_{\cdots}}{abn}$$

$$SS_B = \sum_{j=1}^{b}\frac{y^2_{\cdot j\cdot}}{an} - \frac{y^2_{\cdots}}{abn}$$

$$SS_{AB} = \sum_{i=1}^{a}\sum_{j=1}^{b}\frac{y^2_{ij\cdot}}{n} - \frac{y^2_{\cdots}}{abn} - SS_A - SS_B$$

$$SS_E = SS_T - SS_{AB} - SS_A - SS_B$$

TABLE • ANOVA Table for a Two-Factor Factorial, Fixed-Effects Model

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0
A treatments	SS_A	$a-1$	$MS_A = \frac{SS_A}{a-1}$	$\frac{MS_A}{MS_E}$
B treatments	SS_B	$b-1$	$MS_B = \frac{SS_B}{b-1}$	$\frac{MS_B}{MS_E}$
Interaction	SS_{AB}	$(a-1)(b-1)$	$MS_{AB} = \frac{SS_{AB}}{(a-1)(b-1)}$	$\frac{MS_{AB}}{MS_E}$
Error	SS_E	$ab(n-1)$	$MS_E = \frac{SS_E}{ab(n-1)}$	
Total	SS_T	$abn-1$		

Aircraft Primer Paint Aircraft primer paints are applied to aluminum surfaces by two methods: dipping and spraying. The purpose of using the primer is to improve paint adhesion, and some parts can be primed using either application method. The process engineering group responsible for this operation is interested in learning whether three different primers differ in their adhesion properties. A factorial experiment was performed to investigate the effect of paint primer type and application method on paint adhesion

TABLE • Adhesion Force Data

Primer Type	Dipping	Spraying
1	4.0, 4.5, 4.3	5.4, 4.9, 5.6
2	5.6, 4.9, 5.4	5.8, 6.1, 6.3
3	3.8, 3.7, 4.0	5.5, 5.0, 5.0

Primer Type	Dipping		Spraying		$y_{i\cdot\cdot}$
1	4.0, 4.5, 4.3	12.8	5.4, 4.9, 5.6	15.9	28.7
2	5.6, 4.9, 5.4	15.9	5.8, 6.1, 6.3	18.2	34.1
3	3.8, 3.7, 4.0	11.5	5.5, 5.0, 5.0	15.5	27.0
$y_{\cdot j\cdot}$	40.2		49.6		$89.8 = y_{\cdot\cdot\cdot}$

$$SS_T = \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n} y^2_{ijk} - \frac{y^2_{\cdots}}{abn} = (4.0)^2 + (4.5)^2 + \cdots + (5.0)^2 - \frac{(89.8)^2}{18} = 10.72$$

$$SS_{\text{types}} = \sum_{i=1}^{a} \frac{y_{i\cdot\cdot}^2}{bn} - \frac{y^2_{\cdots}}{abn} = \frac{(28.7)^2 + (34.1)^2 + (27.0)^2}{6} - \frac{(89.8)^2}{18} = 4.58$$

$$SS_{\text{methods}} = \sum_{j=1}^{b} \frac{y^2_{\cdot j\cdot}}{an} - \frac{y^2_{\cdots}}{abn} = \frac{(40.2)^2 + (49.6)^2}{9} - \frac{(89.8)^2}{18} = 4.91$$

$$SS_{\text{interaction}} = \sum_{i=1}^{a}\sum_{j=1}^{b} \frac{y_{ij\cdot}^2}{n} - \frac{y^2_{\cdots}}{abn} - SS_{\text{types}} - SS_{\text{methods}} = \frac{(12.8)^2 + (15.9)^2 + (11.5)^2 + (15.9)^2 + (18.2)^2 + (15.5)^2}{3}$$

$$- \frac{(89.8)^2}{18} - 4.58 - 4.91 = 0.24$$

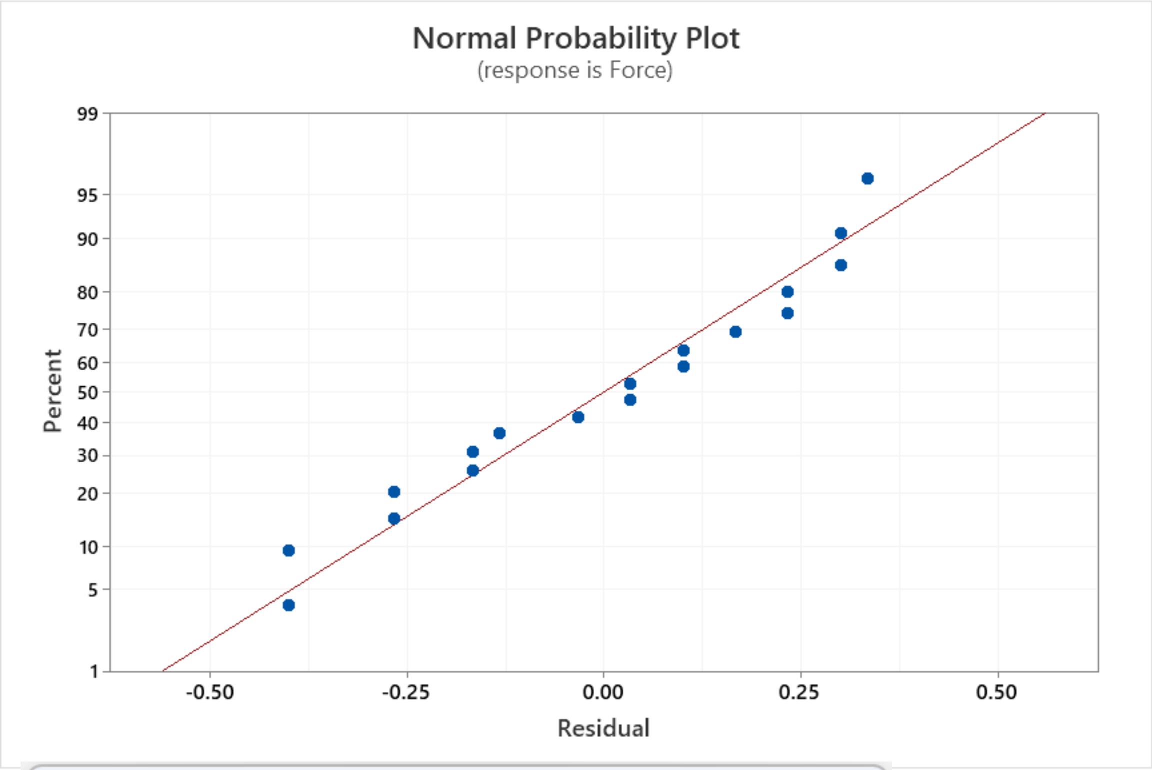
$$SS_E = SS_T - SS_{\text{types}} - SS_{\text{methods}} - SS_{\text{interaction}} = 10.72 - 4.58 - 4.91 - 0.24 = 0.99$$

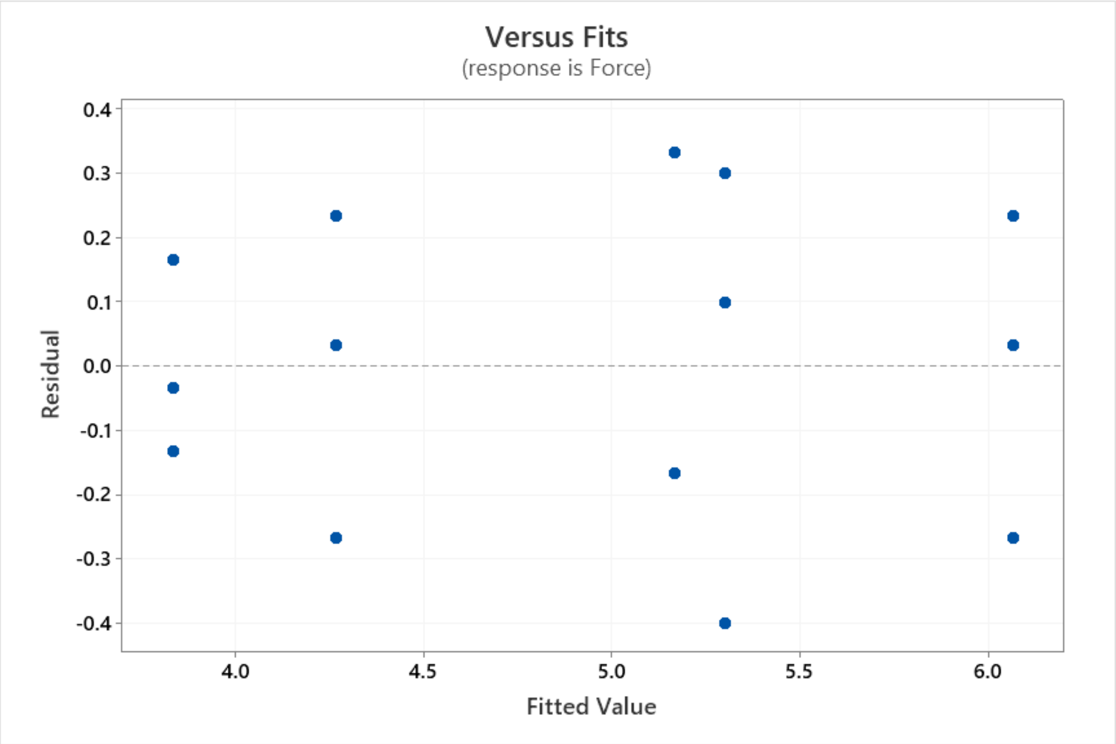
Factor Information

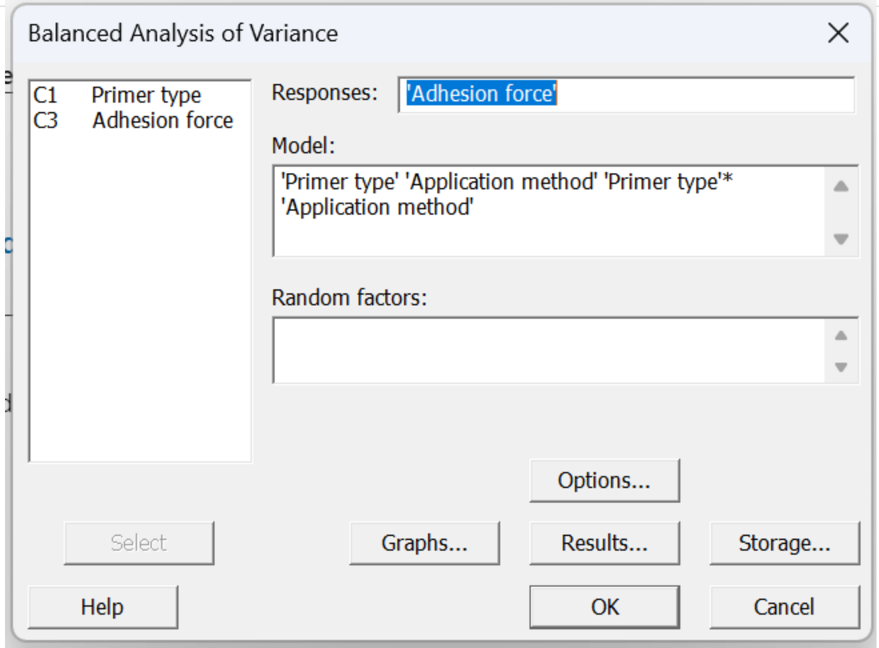
Factor	Type	Levels	Values
Method	Fixed	2	1, 2
Primer type	Fixed	3	1, 2, 3

Analysis of Variance for Force

Source	DF	SS	MS	F	P
Method	1	4.9089	4.90889	59.70	0.000
Primer type	2	4.5811	2.29056	27.86	0.000
Method*Primer	2	0.2411	0.12056	1.47	0.269
Error	12	0.9867	0.08222		
Total	17	10.7178			


Normal Probability Plot
(response is Force)
Percent
Residual
99
95
90
80
70
60
50
40
30
20
10
5
1
-0.50
-0.25
0.00
0.25
0.50


Versus Fits
(response is Force)
Residual
Fitted Value
0.4
0.3
0.2
0.1
0.0
-0.1
-0.2
-0.3
-0.4
4.0
4.5
5.0
5.5
6.0


Balanced Analysis of Variance
C1 Primer type
C3 Adhesion force
Responses: 'Adhesion force'
Model:
'Primer type' 'Application method' 'Primer type'* 'Application method'
Random factors:
Options...
Select
Graphs...
Results...
Storage...
Help
OK
Cancel

Example: An engineer suspects that the surface finish of metal parts is influenced by the type of paint used and the drying time. He selected three drying times: 20, 25, and 30 minutes and used two types of paint. Three parts are tested with each combination of paint type and drying time. The data are as follow

Paint	Drying Time (min) 20	25	30
1	74	73	78
	64	61	85
	50	44	92
2	92	98	66
	86	73	45
	68	88	85

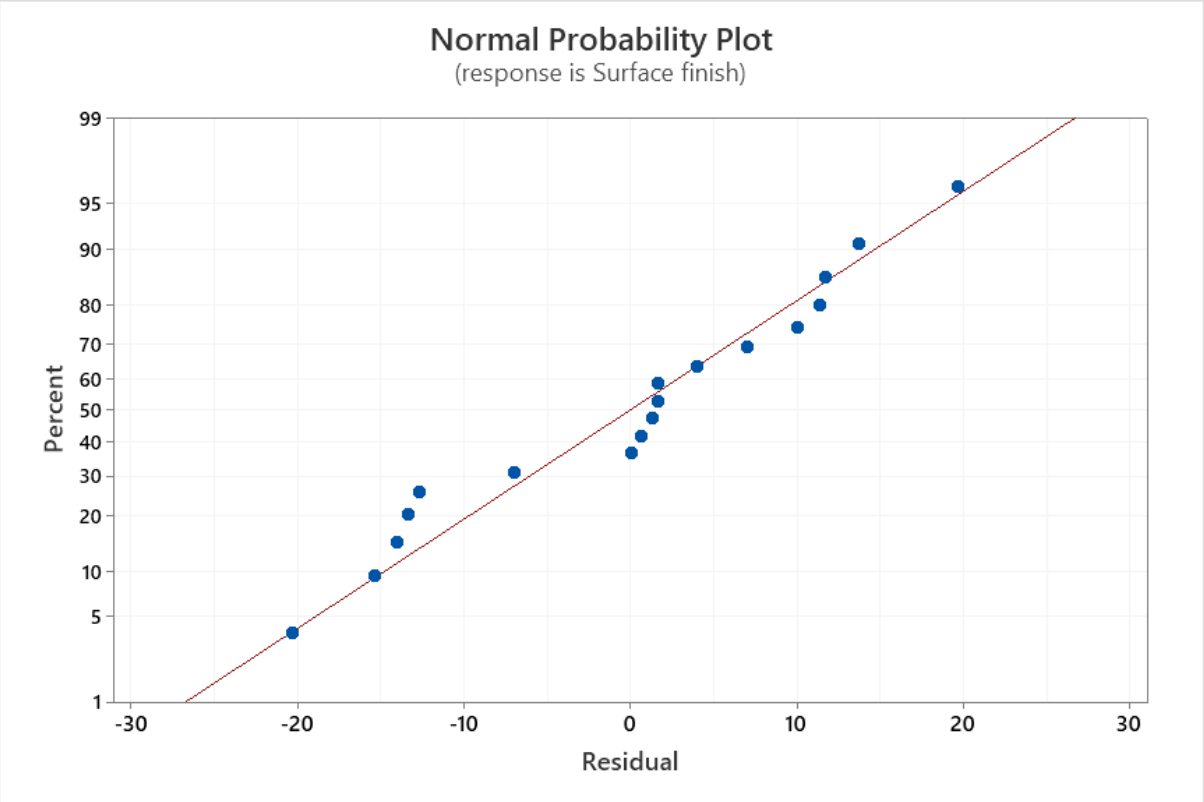
(a) State the hypotheses of interest in this experiment.

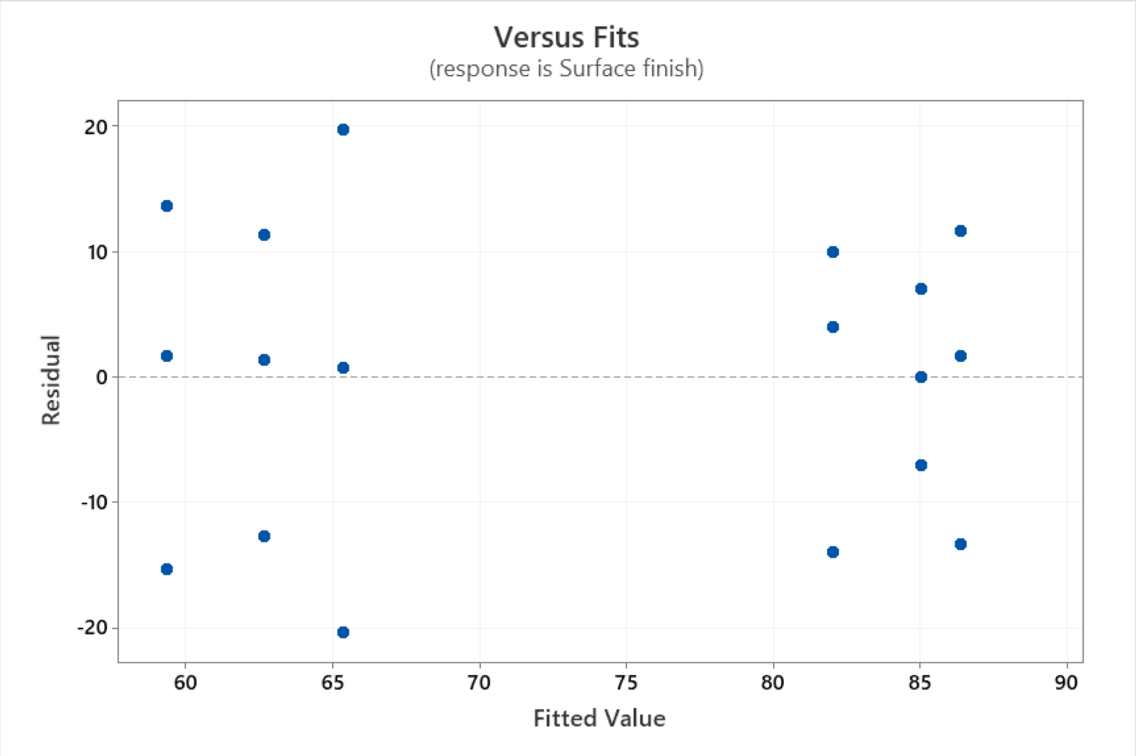
(b) Test the hypotheses in part (a) and draw conclusions using the analysis of variance with $\alpha = 0.05$.

(c) Analyze the residuals from this experiment

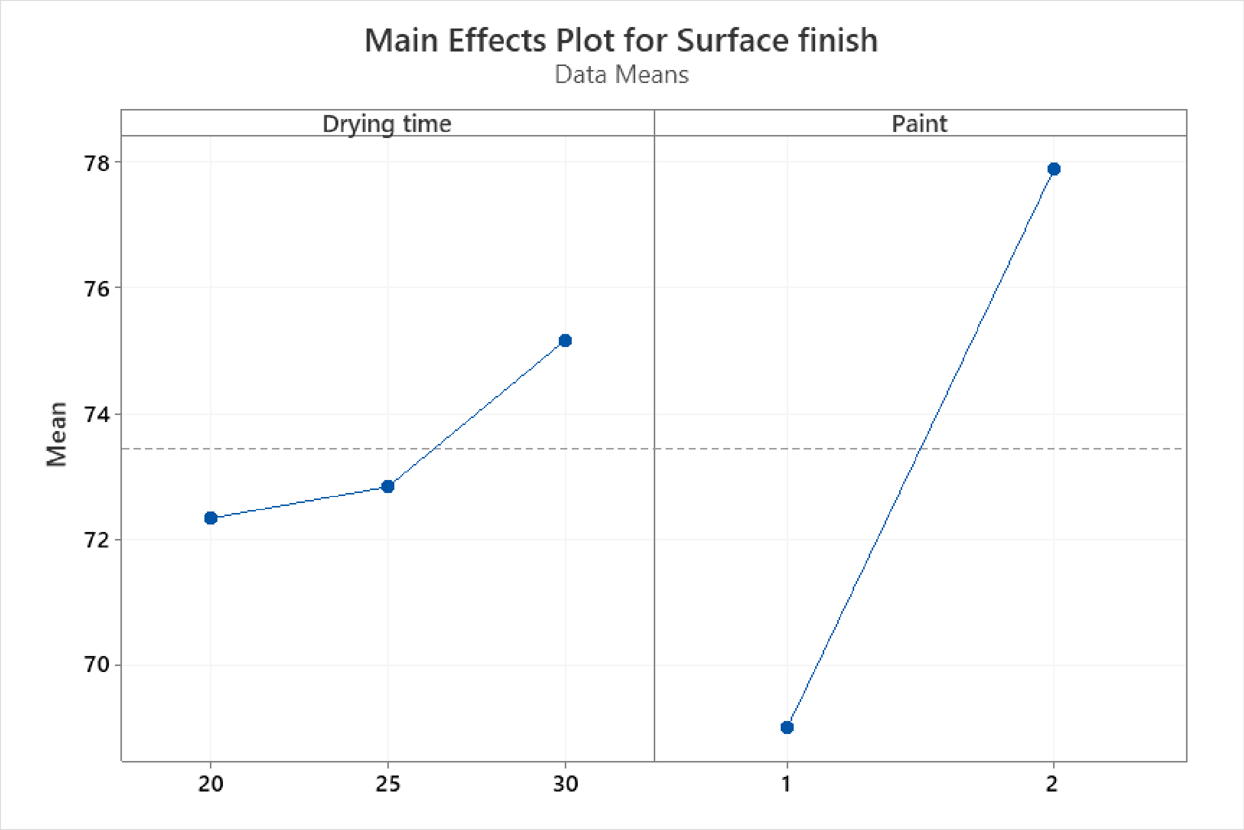
Analysis of Variance for Surface finish

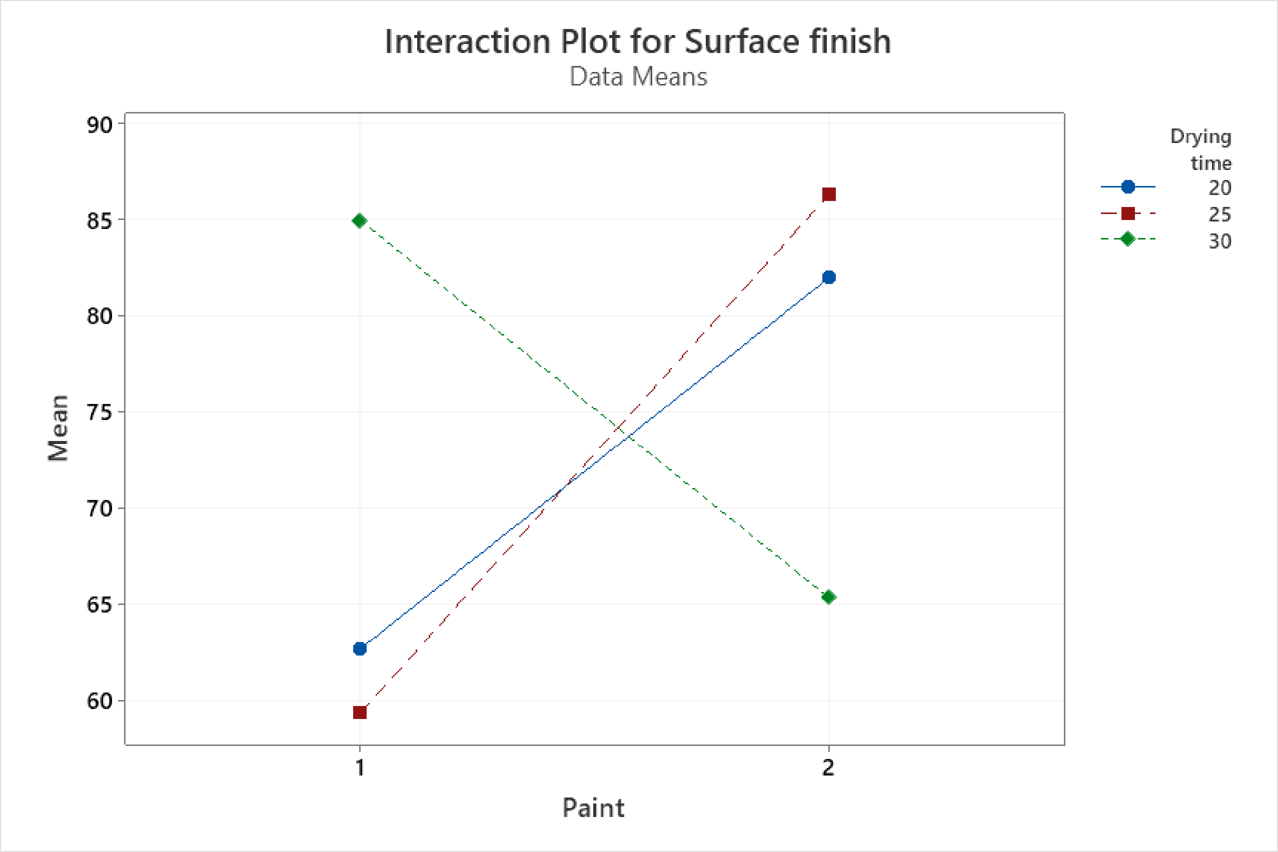
Source	DF	SS	MS	F	P
Drying time	2	27.44	13.72	0.07	0.930
Paint	1	355.56	355.56	1.90	0.193
Drying time*Paint	2	1878.78	939.39	5.03	0.026
Error	12	2242.67	186.89		
Total	17	4504.44			





Model Adequacy


Main Effects Plot for Surface finish
Data Means
Drying time
Paint
Mean
78
76
74
72
70
20
25
30
1
2


Interaction Plot for Surface finish
Data Means
Drying time
20
25
30
Mean
90
85
80
75
70
65
60
1
2
Paint

Example: the effects of cyclic loading frequency and environment conditions on fatigue crack growth at a constant 22 MPa stress for a particular material. The data follow. The response variable is fatigue crack growth rate.

(a) Is there indication that either factor affects crack growth rate? Is there any indication of interaction? Use $\alpha = 0.05$.

(b) Analyze the residuals from this experiment.

(c) Repeat the analysis in part (a) using ln(y) as the response. Analyze the residuals from this new response variable and comment on the results.

		Environment		
		Air	**H_2O**	**Salt H_2O**
Frequency	10	2.29 2.47 2.48 2.12	2.06 2.05 2.23 2.03	1.90 1.93 1.75 2.06
	1	2.65 2.68 2.06 2.38	3.20 3.18 3.96 3.64	3.10 3.24 3.98 3.24
	0.1	2.24 2.71 2.81 2.08	11.00 11.00 9.06 11.30	9.96 10.01 9.36 10.40

General Factorial Experiments

Many experiments involve more than two factors. In this section, we introduce the case in which there are a levels of factor A, b levels of factor B, c levels of factor C, and so on, arranged in a factorial experiment. In general, there are $a \times b \times c \cdots \times n$ total observations if there are n replicates of the complete experiment.

For example, consider the **three-factor-factorial experiment**, with underlying model

$$Y_{ijkl} = \mu + \tau_i + \beta_j + \gamma_k + (\tau\beta)_{ij} + (\tau\gamma)_{ik} + (\beta\gamma)_{jk} + (\tau\beta\gamma)_{ijk} + \epsilon_{ijkl} \begin{cases} i = 1,2,\ldots,a \\ j = 1,2,\ldots,b \\ k = 1,2,\ldots,c \\ l = 1,2,\ldots,n \end{cases}$$

Notice that the model contains three main effects, three two-factor interactions, a three-factor interaction, and an error term. Assuming that A, B, and C are fixed factors .
Note that there must be at least two replicates ($n \geq 2$) to compute an error sum of squares.
The F-test on main effects and interactions follows directly from the expected mean squares.
These ratios follow F-distributions under the respective null hypotheses.

TABLE • Analysis of Variance Table for the Three-Factor Fixed Effects Model

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	Expected Mean Squares	F_0
A	SS_A	$a-1$	MS_A	$\sigma^2 + \frac{bcn\sum \tau_i^2}{a-1}$	$\frac{MS_A}{MS_E}$
B	SS_B	$b-1$	MS_B	$\sigma^2 + \frac{acn\sum \beta_j^2}{b-1}$	$\frac{MS_B}{MS_E}$
C	SS_C	$c-1$	MS_C	$\sigma^2 + \frac{abn\sum \gamma_k^2}{c-1}$	$\frac{MS_C}{MS_E}$
AB	SS_{AB}	$(a-1)(b-1)$	MS_{AB}	$\sigma^2 + \frac{cn\sum\sum (\tau\beta)_{ij}^2}{(a-1)(b-1)}$	$\frac{MS_{AB}}{MS_E}$
AC	SS_{AC}	$(a-1)(c-1)$	MS_{AC}	$\sigma^2 + \frac{bn\sum\sum (\tau\gamma)_{ik}^2}{(a-1)(c-1)}$	$\frac{MS_{AC}}{MS_E}$
BC	SS_{BC}	$(b-1)(c-1)$	MS_{BC}	$\sigma^2 + \frac{cn\sum\sum (\beta\gamma)_{jk}^2}{(b-1)(c-1)}$	$\frac{MS_{BC}}{MS_E}$
ABC	SS_{ABC}	$(a-1)(b-1)(c-1)$	MS_{ABC}	$\sigma^2 + \frac{n\sum\sum\sum (\tau\beta\gamma)_{ijk}^2 n}{(a-1)(b-1)(c-1)}$	$\frac{MS_{ABC}}{MS_E}$
Error	SS_E	$abc(n-1)$	MS_E	σ^2	
Total	SS_T	$abcn-1$			

Example 14-2 **Surface Roughness** A mechanical engineer is studying the surface roughness of a part produced in a metal-cutting operation. Three factors, feed rate (A), depth of cut (B), and tool angle (C), are of interest. All three factors have been assigned two levels, and two replicates of a factorial design are run.

TABLE • Coded Surface Roughness Data

	Depth of Cut (B)				
	0.025 inch		0.040 inch		
	Tool Angle (C)		Tool Angle (C)		
Feed Rate (A)	15°	25°	15°	25°	$y_{i\cdots}$
	9	11	9	10	
20 inches per minute	7	10	11	8	75
	10	10	12	16	
30 inches per minute	12	13	15	14	102

ANOVA

Factor	Type	Levels	Values	
Feed	fixed	2	20	30
Depth	fixed	2	0.025	0.040
Angle	fixed	2	15	25

Analysis of Variance for Roughness

Source	DF	SS	MS	F	P
Feed	1	45.563	45.563	18.69	0.003
Depth	1	10.563	10.563	4.33	0.071
Angle	1	3.063	3.063	1.26	0.295
Feed*Depth	1	7.563	7.563	3.10	0.116
Feed*Angle	1	0.062	0.062	0.03	0.877
Depth*Angle	1	1.563	1.563	0.64	0.446
Feed*Depth*Angle	1	5.062	5.062	2.08	0.188
Error	8	19.500	2.437		
Total	15	92.938			

Example: The quality control department of a fabric finishing plant is studying the effects of several factors on dyeing for a blended cotton/synthetic cloth used to manufacture shirts. Three operators, three cycle times, and two temperatures were selected, and three small specimens of cloth were dyed under each set of conditions. The finished cloth was compared to a standard, and a numerical score was assigned. The results are shown in the following table.

(a) State and test the appropriate hypotheses using the analysis of variance with $\alpha = 0.05$.

(b) Graphically analyze the residuals from this experiment.

	Temperature					
	300°			350°		
	Operator			Operator		
Cycle Time	**1**	**2**	**3**	**1**	**2**	**3**
	23	27	31	24	38	34
40	24	28	32	23	36	36
	25	26	28	28	35	39
	36	34	33	37	34	34
50	35	38	34	39	38	36
	36	39	35	35	36	31
	28	35	26	26	36	28
60	24	35	27	29	37	26
	27	34	25	25	34	34

Analysis of Variance

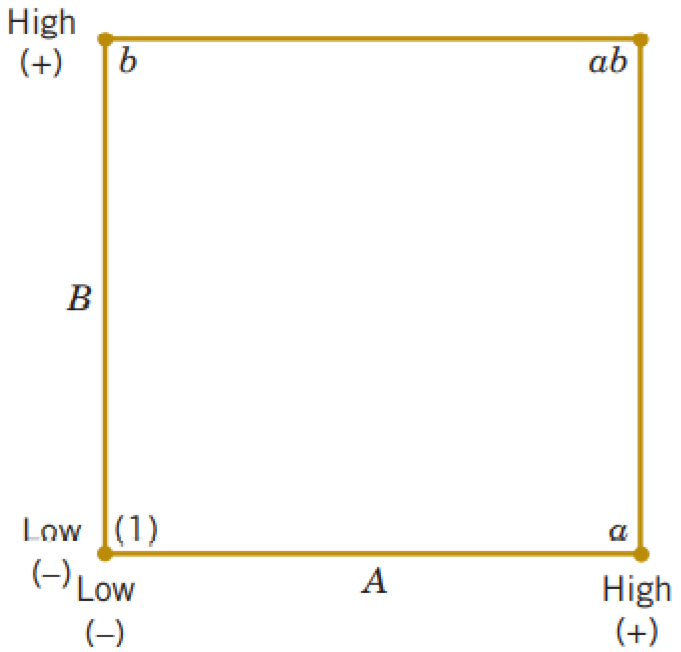
Source	DF	Adj SS	Adj MS	F-Value	P-Value
Cycle time	2	473.59	236.796	41.38	0.000
Operator	2	174.48	87.241	15.25	0.000
Temperature	1	39.19	39.185	6.85	0.013
Cycle time*Operator	4	230.96	57.741	10.09	0.000
Cycle time*Temperature	2	94.70	47.352	8.28	0.001
Operator*Temperature	2	11.59	5.796	1.01	0.373
Cycle time*Operator*Temperature	4	94.52	23.630	4.13	0.007
Error	36	206.00	5.722		
Total	53	1325.04			

2^k Factorial Designs

Factorial designs are frequently used in experiments involving several factors where it is necessary to study the joint effect of the factors on a response.

2^2 DESIGN

treatment combination ***a*** indicates that factor A is at the high level and factor B is at the low level. The treatment combination with both factors at the low level is represented by (1).



Treatment	A	B
(1)	–	–
a	+	–
b	–	+
ab	+	+

FIGURE The 2^2 factorial design.

TABLE • Signs for Effects in the 2^2 Design

Treatment Combination	Factorial Effect			
	I	A	B	AB
(1)	+	−	−	+
a	+	+	−	−
b	+	−	+	−
ab	+	+	+	+

Main Effect of Factor $A: 2^2$ Design

$$A = \bar{y}_{A+} - \bar{y}_{A-} = \frac{a+ab}{2n} - \frac{b+(1)}{2n} = \frac{1}{2n}\left[a+ab-b-(1)\right]$$

→ Contrast A

Main Effect of Factor $B: 2^2$ Design

$$B = \bar{y}_{B+} - \bar{y}_{B-} = \frac{b+ab}{2n} - \frac{a+(1)}{2n} = \frac{1}{2n}\left[b+ab-a-(1)\right]$$

→ Contrast B

Interaction Effect $AB: 2^2$ Design

$$AB = \frac{ab+(1)}{2n} - \frac{a+b}{2n} = \frac{1}{2n}\left[ab+(1)-a-b\right]$$

→ Contrast AB

Relationship Between a Contrast and an Effect

$$\text{Effect} = \frac{\text{Contrast}}{n2^{k-1}}$$

Sum of Squares for an Effect

$$SS = \frac{(\text{Contrast})^2}{n2^k}$$

Example: A basic processing step in this industry is to grow an epitaxial layer on polished silicon wafers. The wafers are mounted on a susceptor and positioned inside a bell jar. Chemical vapors are introduced through nozzles near the top of the jar. The susceptor is rotated, and heat is applied. These conditions are maintained until the epitaxial layer is thick enough.

A = deposition time and B = arsenic flow rate.

The two levels of deposition time are − = short and + = long, and the two levels of arsenic flow rate are − = 55% and + = 59%

TABLE • The 2^2 Design for the Epitaxial Process Experiment

Treatment Combination	Design Factors			Thickness (μm)				Thickness (μm)	
	A	*B*	*AB*					Total	Average
(1)	−	−	+	14.037	14.165	13.972	13.907	56.081	14.020
a	+	−	−	14.821	14.757	14.843	14.878	59.299	14.825
b	−	+	−	13.880	13.860	14.032	13.914	55.686	13.922
ab	+	+	+	14.888	14.921	14.415	14.932	59.156	14.789

$$A = \frac{1}{2n}\left[a + ab - b - (1)\right] = \frac{1}{2(4)}\left[59.299 + 59.156 - 55.686 - 56.081\right] = 0.836$$

$$B = \frac{1}{2n}\left[b + ab - a - (1)\right] = \frac{1}{2(4)}\left[55.686 + 59.156 - 59.299 - 56.081\right] = -0.067$$

$$AB = \frac{1}{2n}\left[ab + (1) - a - b\right]$$

$$AB = \frac{1}{2(4)}\left[59.156 + 56.081 - 59.299 - 55.686\right] = 0.032$$

$$SS_A = \frac{\left[a + ab - b - (1)\right]^2}{16} = \frac{\left[6.688\right]^2}{16} = 2.7956$$

$$SS_B = \frac{\left[b + ab - a - (1)\right]^2}{16} = \frac{\left[-0.538\right]^2}{16} = 0.0181$$

$$SS_{AB} = \frac{\left[ab + (1) - a - b\right]^2}{16} = \frac{\left[0.252\right]^2}{16} = 0.0040$$

$$SS_T = 14.037^2 + \cdots + 14.932^2 - \frac{(56.081 + \cdots + 59.156)^2}{16} = 3.0672$$

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	f_0	*P*-Value
A (deposition time)	2.7956	1	2.7956	134.40	7.07 E-8
B (arsenic flow)	0.0181	1	0.0181	0.87	0.38
AB	0.0040	1	0.0040	0.19	0.67
Error	0.2495	12	0.0208		
Total	3.0672	15			

Models and Residual Analysis

It is easy to obtain a model for the response and residuals from a 2^k design by fitting a **regression model** to the data. For the epitaxial process experiment, the regression model is

$$Y = \beta_0 + \beta_1 x_1 + \epsilon$$

because the only active variable is deposition time, which is represented by a coded variable x_1. The low and high levels of deposition time are assigned values $x_1 = -1$ and $x_1 = +1$, respectively. The least squares fitted model is

$$\hat{y} = 14.389 + \left(\frac{0.836}{2}\right)x_1$$

where the intercept $\hat{\beta}_0$ is the grand average of all 16 observations ($\bar{y}$) and the slope $\hat{\beta}_1$ is one-half the effect estimate for deposition time. The regression coefficient is one-half the effect estimate because regression coefficients measure the effect of a unit change in x_1 on the mean of Y, and the effect estimate is based on a two-unit change from -1 to $+1$.

Coefficient and Effect

$$\hat{\beta} = \frac{\text{effect}}{2} = \frac{\bar{y}_+ - \bar{y}_-}{2}$$

Standard Error of a Coefficient

$$\text{Standard error } \hat{\beta} = \frac{\hat{\sigma}}{2}\sqrt{\frac{1}{n2^{k-1}} + \frac{1}{n2^{k-1}}} = \hat{\sigma}\sqrt{\frac{1}{n2^k}}$$

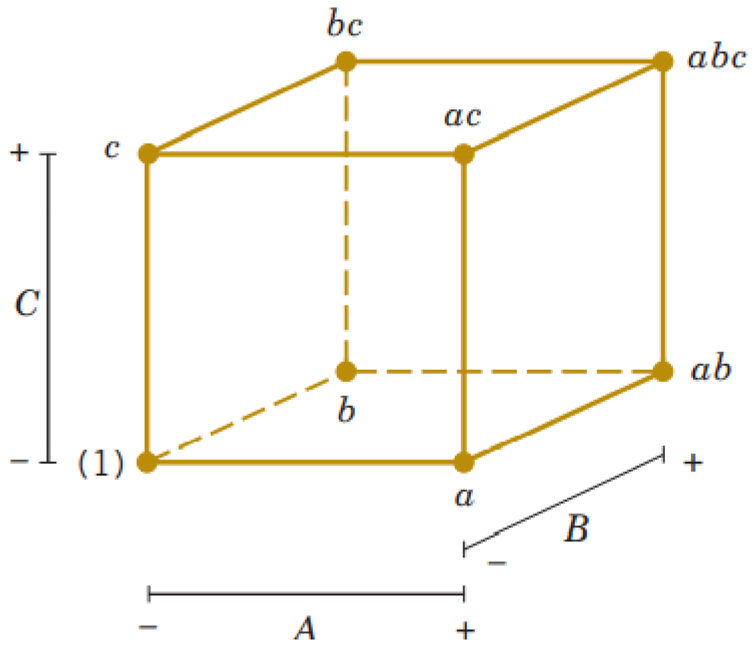
t-statistic for a Coefficient

$$t = \frac{\hat{\beta}}{\text{Standard error } \hat{\beta}} = \frac{(\bar{y}_+ - \bar{y}_-)/2}{\hat{\sigma}\sqrt{\frac{1}{n2^k}}} \qquad (14\text{-}18)$$

TABLE • Analysis for the Epitaxial Process Experiment

Term	Effect	Coefficient	SE Coefficient	t	P-Value
Constant	14.3889	0.03605	399.17	0.000	
A	0.8360	0.4180	0.03605	11.60	0.000
B	–0.0672	–0.0336	0.03605	–0.93	0.369
AB	0.0315	0.0157	0.03605	0.44	0.670

2^k DESIGN FOR $k \geq 3$ FACTORS



(a) Geometric view

Run	A	B	C
1	−	−	−
2	+	−	−
3	−	+	−
4	+	+	−
5	−	−	+
6	+	−	+
7	−	+	+
8	+	+	+

(b) 2^3 design matrix

TABLE • Algebraic Signs for Calculating Effects in the 2^3 Design

Treatment Combination	Factorial Effect							
	I	A	B	AB	C	AC	BC	ABC
(1)	+	−	−	+	−	+	+	−
a	+	+	−	−	−	−	+	+
b	+	−	+	−	−	+	−	+
ab	+	+	+	+	−	−	−	−
c	+	−	−	+	+	−	−	+
ac	+	+	−	−	+	+	−	−
bc	+	−	+	−	+	−	+	−
abc	+	+	+	+	+	+	+	+

Main Effect of Factor $A: 2^3$ Design

$$A = \bar{y}_{A+} - \bar{y}_{A-} = \frac{1}{4n}\left[a + ab + ac + abc - (1) - b - c - bc\right]$$

Main Effect of Factor $B: 2^3$ Design

$$B = \bar{y}_{B+} - \bar{y}_{B-} = \frac{1}{4n}\left[b + ab + bc + abc - (1) - a - c - ac\right]$$

Main Effect of Factor $C: 2^3$ Design

$$C = \bar{y}_{C+} - \bar{y}_{C-} = \frac{1}{4n}\left[c + ac + bc + abc - (1) - a - b - ab\right]$$

Two-Factor Interaction Effects: 2^3 Design

$$AB = \frac{1}{4n}\left[abc - bc + ab - b - ac + c - a + (1)\right]$$

Two-Factor Interaction Effect: 2^3 Design

$$AC = \frac{1}{4n}\left[(1) - a + b - ab - c + ac - bc + abc\right]$$

$$BC = \frac{1}{4n}\left[(1) + a - b - ab - c - ac + bc + abc\right]$$

Three-Factor Interaction Effect: 2^3 Design

$$ABC = \frac{1}{4n}\left[abc - bc - ac + c - ab + b + a - (1)\right]$$

Relationship Between a Contrast and an Effect

$$\text{Effect} = \frac{\text{Contrast}}{n2^{k-1}}$$

Coefficient and Effect

$$\hat{\beta} = \frac{\text{effect}}{2} = \frac{\bar{y}_+ - \bar{y}_-}{2}$$

Sum of Squares for an Effect

$$SS = \frac{(\text{Contrast})^2}{n2^k}$$

Standard Error of a Coefficient

$$\text{Standard error } \hat{\beta} = \frac{\hat{\sigma}}{2}\sqrt{\frac{1}{n2^{k-1}} + \frac{1}{n2^{k-1}}} = \hat{\sigma}\sqrt{\frac{1}{n2^k}}$$

TABLE • Coded Surface Roughness Data

	Depth of Cut (B)				
	0.025 inch		0.040 inch		
	Tool Angle (C)		Tool Angle (C)		
Feed Rate (A)	15°	25°	15°	25°	$y_{i\cdots}$
	9	11	9	10	
20 inches per minute	7	10	11	8	75
	10	10	12	16	
30 inches per minute	12	13	15	14	102

Treatment Combinations	Design Factors A	B	C	Surface Roughness	Totals
(1)	–1	–1	–1	9, 7	16
a	1	–1	–1	10, 12	22
b	–1	1	–1	9, 11	20
ab	1	1	–1	12, 15	27
c	–1	–1	1	11, 10	21
ac	1	–1	1	10, 13	23
bc	–1	1	1	10, 8	18
abc	1	1	1	16, 14	30

$$SS_A = \frac{(\text{contrast}_A)^2}{n2^k} = \frac{(27)^2}{2(8)} = 45.5625$$

It is easy to verify that the other effects are

$$\begin{aligned}
B &= 1.625 \\
C &= 0.875 \\
AB &= 1.375 \\
AC &= 0.125 \\
BC &= -0.625 \\
ABC &= 1.125
\end{aligned}$$

Term	Effect	Coefficient	SE Coefficient	t	P-Value
Constant		11.0625	0.3903	28.34	0.000
A	3.3750	1.6875	0.3903	4.32	0.003
B	1.6250	0.8125	0.3903	2.08	0.071
C	0.8750	0.4375	0.3903	1.12	0.295
AB	1.3750	0.6875	0.3903	1.76	0.116
AC	0.1250	0.0625	0.3903	0.16	0.877
BC	–0.6250	–0.3125	0.3903	–0.80	0.446
ABC	1.1250	0.5625	0.3903	1.44	0.188

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	f_0	P-Value
A	45.5625	1	45.5625	18.69	0.0025
B	10.5625	1	10.5625	4.33	0.0709
C	3.0625	1	3.0625	1.26	0.2948
AB	7.5625	1	7.5625	3.10	0.1162
AC	0.0625	1	0.0625	0.03	0.8784
BC	1.5625	1	1.5625	0.64	0.4548
ABC	5.0625	1	5.0625	2.08	0.1875
Error	19.5000	8	2.4375		
Total	92.9375	15			

TABLE • 14-18 Computer Analysis for the Surface Roughness Experiment in Example 14-4

Estimated Effects and Coefficients for Roughness

Term	Effect	Coef	StDev Coef	T	P
Constant		11.0625	0.3903	28.34	0.000
Feed	3.3750	1.6875	0.3903	4.32	0.003
Depth	1.6250	0.8125	0.3903	2.08	0.071
Angle	0.8750	0.4375	0.3903	1.12	0.295
Feed*Depth	1.3750	0.6875	0.3903	1.76	0.116
Feed*Angle	0.1250	0.0625	0.3903	0.16	0.877
Depth*Angle	–0.6250	–0.3125	0.3903	–0.80	0.446
Feed*Depth*Angle	1.1250	0.5625	0.3903	1.44	0.188

Analysis of Variance for Roughness

Source	DF	Seq SS	Adj SS	Adj MS	F	P
Main effects	3	59.188	59.188	19.729	8.09	0.008
2-Way interactions	3	9.187	9.187	3.062	1.26	0.352
3-Way interactions	1	5.062	5.062	5.062	2.08	0.188
Residual error	8	19.500	19.500	2.437		
Pure error	8	19.500	19.500	2.437		
Total	15	92.938				

Models and Residual Analysis

We may obtain the residuals from a 2^k design by using the method demonstrated earlier for the 2^2 design. As an example, consider the surface roughness experiment. The three largest effects are *A*, *B*, and the *AB* interaction. The regression model used to obtain the predicted values is

$$Y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{12} x_1 x_2 + \epsilon$$

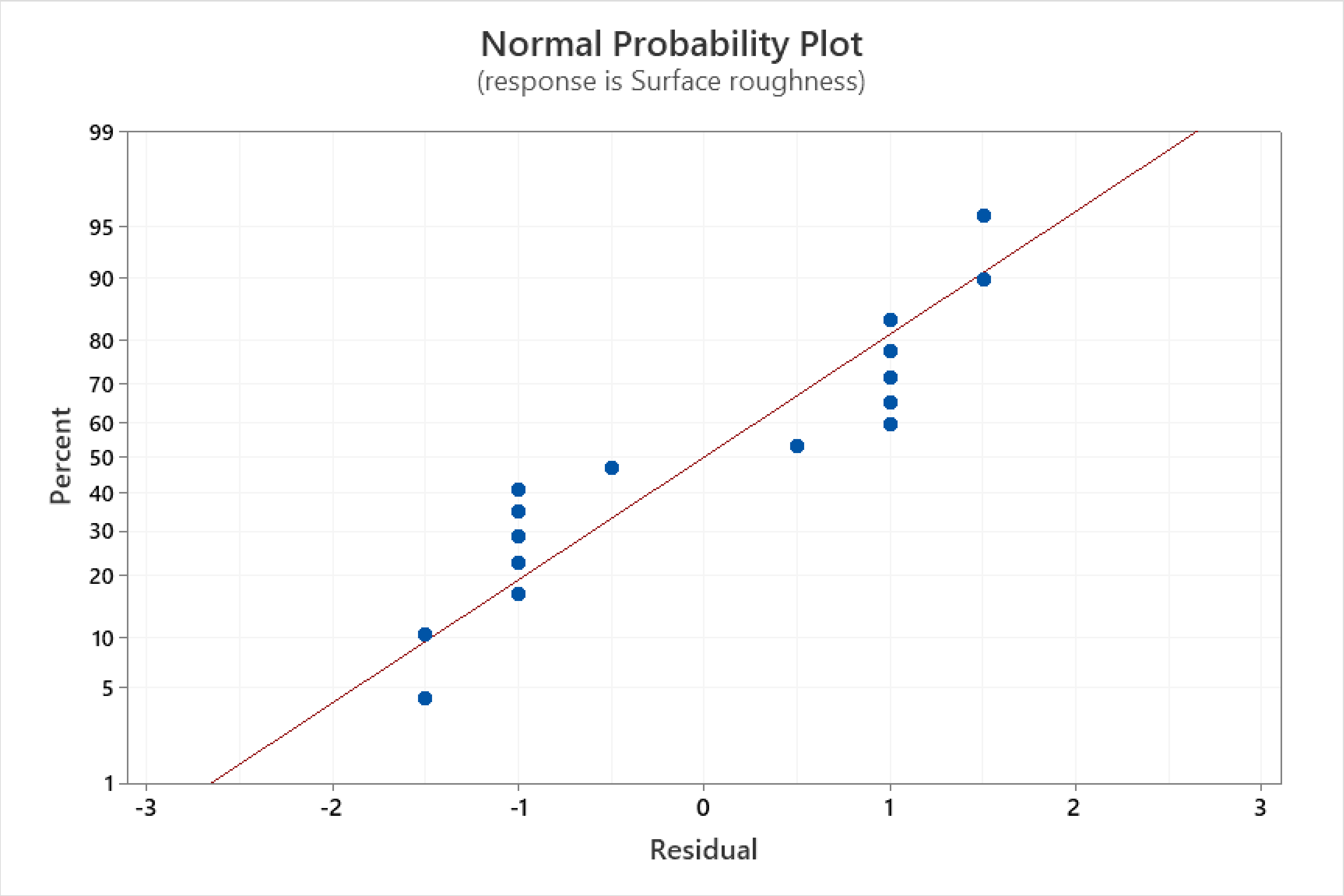
where x_1 represents factor *A*, x_2 represents factor *B*, and $x_1 x_2$ represents the *AB* interaction. The regression coefficients β_1, β_2, and β_{12} are estimated by one-half the corresponding effect estimates, and β_0 is the grand average. Thus,

$$\begin{aligned}\hat{y} &= 11.0625 + \left(\frac{3.375}{2}\right) x_1 + \left(\frac{1.625}{2}\right) x_2 + \left(\frac{1.375}{2}\right) x_1 x_2 \\ &= 11.0625 + 1.6875 x_1 + 0.8125 x_2 + 0.6875 x_1 x_2\end{aligned}$$

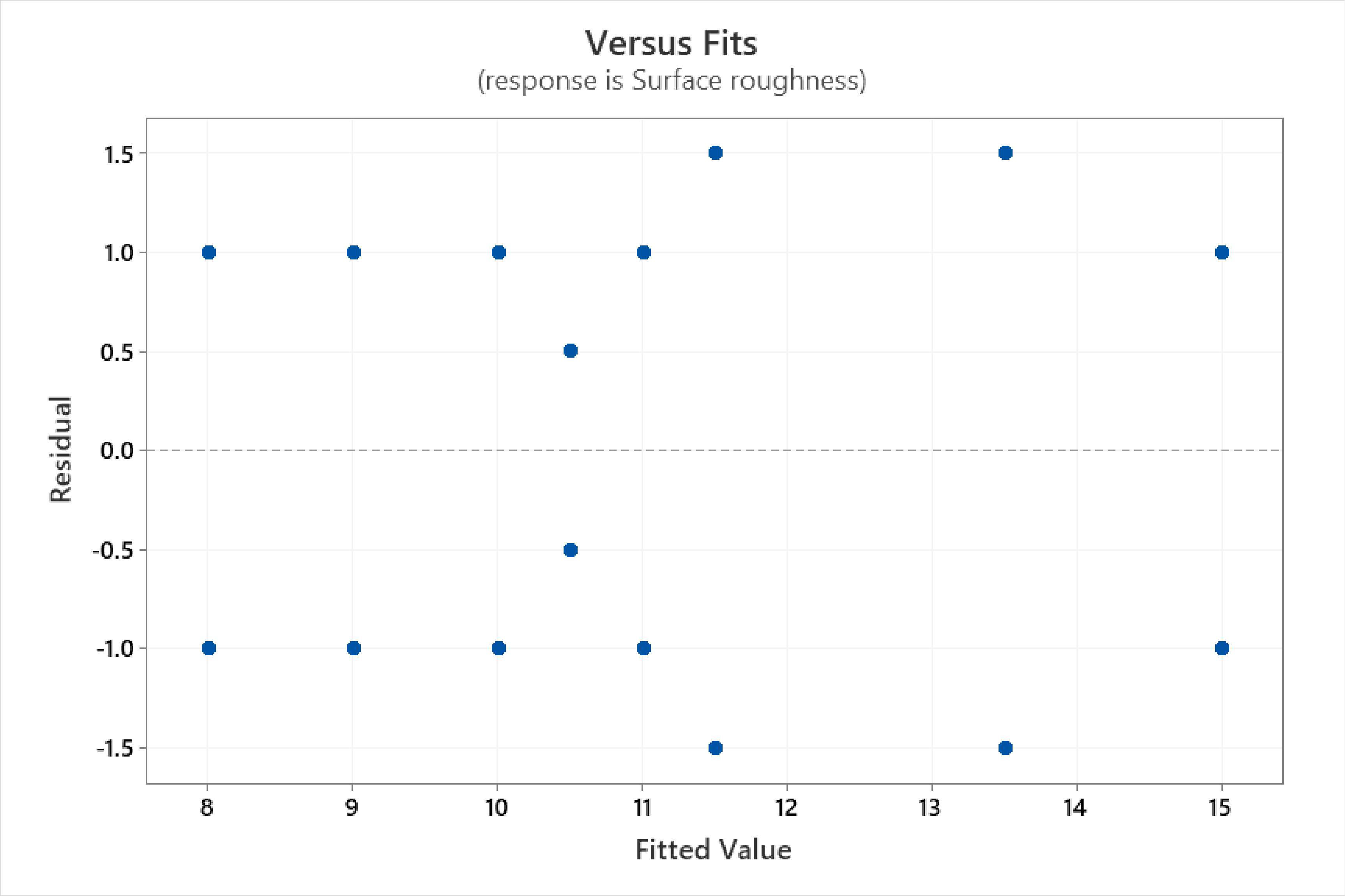
Note that the regression coefficients are presented in the upper panel of Table 14-18. The predicted values would be obtained by substituting the low and high levels of *A* and *B* into this equation. To illustrate this, at the treatment combination where *A*, *B*, and *C* are all at the low level, the predicted value is

$$\hat{y} = 11.0625 + 1.6875(-1) + 0.8125(-1) + 0.6875(-1)(-1) = 9.25$$

Because the observed values at this run are 9 and 7, the residuals are $9 - 9.25 = -0.25$ and $7 - 9.25 = -2.25$. Residuals for the other 14 runs are obtained similarly.



Normal Probability Plot
(response is Surface roughness)
Percent
99
95
90
80
70
60
50
40
30
20
10
5
1
-3
-2
-1
0
1
2
3
Residual


Versus Fits
(response is Surface roughness)
Residual
Fitted Value
1.5
1.0
0.5
0.0
-0.5
-1.0
-1.5
8
9
10
11
12
13
14
15

Single Replicate of the 2^k Design

- As the number of factors in a factorial experiment increases, the number of effects that can be estimated also increases.
    - For example, a 2^4 experiment has 4 main effects, 6 two-factor interactions, 4 three-factor interactions, and 1 four-factor interaction, and a 2^6 experiment has 6 main effects, 15 two-factor interactions, 20 three-factor interactions, 15 four-factor interactions, 6 five-factor interactions, and 1 six-factor interaction.
- In most situations, the **sparsity of effects principle** applies; that is, the system is usually dominated by the main effects and low-order interactions. The three-factor and higher order interactions are usually negligible. Therefore, when the number of factors is moderately large, say, $k \geq 4$ or 5, a common practice is to run only a single replicate of the 2^k design and then pool or combine the higher order interactions as an estimate of error. Sometimes a single replicate of a 2^k design is called an **unreplicated 2^k factorial design**

Example 14-5 **Plasma Etch** An article in *Solid State Technology* ["Orthogonal Design for Process Optimization and Its Application in Plasma Etching" (May 1987, pp. 127–132)] describes the application of factorial designs in developing a nitride etch process on a single-wafer plasma etcher. The process uses C_2F_6 as the reactant gas. It is possible to vary the gas flow, the power applied to the cathode, the pressure in the reactor chamber, and the spacing between the anode and the cathode (gap). Several response variables would usually be of interest in this process, but in this example, we concentrate on etch rate for silicon nitride.

We use a single replicate of a 2^4 design to investigate this process. Because it is unlikely that the three- and four-factor interactions are significant, we tentatively plan to combine them as an estimate of error. The factor levels used in the design follow:

Design Factor

Level	Gap (cm)	Pressure (mTorr)	C_2F_6 Flow (SCCM)	Power (w)
Low (–)	0.80	450	125	275
High (+)	1.20	550	200	325

TABLE • 14-19 The 2^4 Design for the Plasma Etch Experiment

A (Gap)	*B* (Pressure)	*C* (C_2F_6 Flow)	*D* (Power)	Etch Rate (Å/min)
–1	–1	–1	–1	550
1	–1	–1	–1	669
–1	1	–1	–1	604
1	1	–1	–1	650
–1	–1	1	–1	633
1	–1	1	–1	642
–1	1	1	–1	601
1	1	1	–1	635
–1	–1	–1	1	1037
1	–1	–1	1	749
–1	1	–1	1	1052
1	1	–1	1	868
–1	–1	1	1	1075
1	–1	1	1	860
–1	1	1	1	1063
1	1	1	1	729

TABLE • 14-20 Contrast Constants for the 2^4 Design

	A	*B*	*AB*	*C*	*AC*	*BC*	*ABC*	*D*	*AD*	*BD*	*ABD*	*CD*	*ACD*	*BCD*	*ABCD*
(1)	–	–	+	–	+	+	–	–	+	+	–	+	–	–	+
a	+	–	–	–	–	+	+	–	–	+	+	+	+	–	–
b	–	+	–	–	+	–	+	–	+	–	+	+	–	+	–
ab	+	+	+	–	–	–	–	–	–	–	–	+	+	+	+
c	–	–	+	+	–	–	+	–	+	+	–	–	+	+	–
ac	+	–	–	+	+	–	–	–	–	+	+	–	–	+	+
bc	–	+	–	+	–	+	–	–	+	–	+	–	+	–	+
abc	+	+	+	+	+	+	+	–	–	–	–	–	–	–	–
d	–	–	+	–	+	+	–	+	–	–	+	–	+	+	–
ad	+	–	–	–	–	+	+	+	+	–	–	–	–	+	+
bd	–	+	–	–	+	–	+	+	–	+	–	–	+	–	+
abd	+	+	+	–	–	–	–	+	+	+	+	–	–	–	–
cd	–	–	+	+	–	–	+	+	–	–	+	+	–	–	+
acd	+	–	–	+	+	–	–	+	+	–	–	+	+	–	–
bcd	–	+	–	+	–	+	–	+	–	+	–	+	–	+	–
abcd	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+

Coded Coefficients

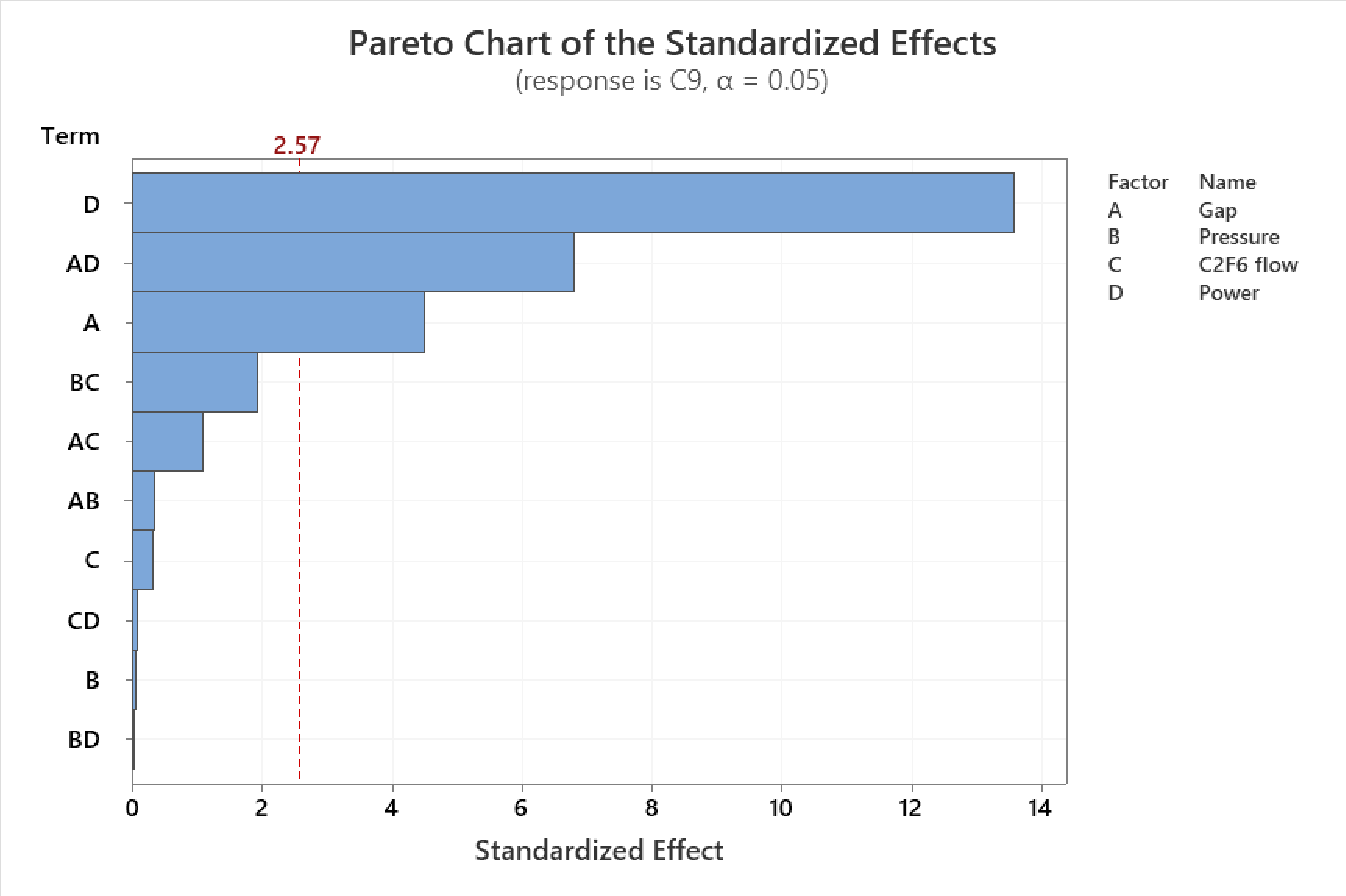
Term	Effect	Coef	SE Coef	T-Value	P-Value	VIF
Constant		776.1	11.3	68.77	0.000	
Gap	-101.6	-50.8	11.3	-4.50	0.006	1.00
Pressure	-1.6	-0.8	11.3	-0.07	0.945	1.00
C2F6 flow	7.4	3.7	11.3	0.33	0.757	1.00
Power	306.1	153.1	11.3	13.56	0.000	1.00
Gap*Pressure	-7.9	-3.9	11.3	-0.35	0.741	1.00
Gap*C2F6 flow	-24.9	-12.4	11.3	-1.10	0.321	1.00
Gap*Power	-153.6	-76.8	11.3	-6.81	0.001	1.00
Pressure*C2F6 flow	-43.9	-21.9	11.3	-1.94	0.109	1.00
Pressure*Power	-0.6	-0.3	11.3	-0.03	0.979	1.00
C2F6 flow*Power	-2.1	-1.1	11.3	-0.09	0.929	1.00

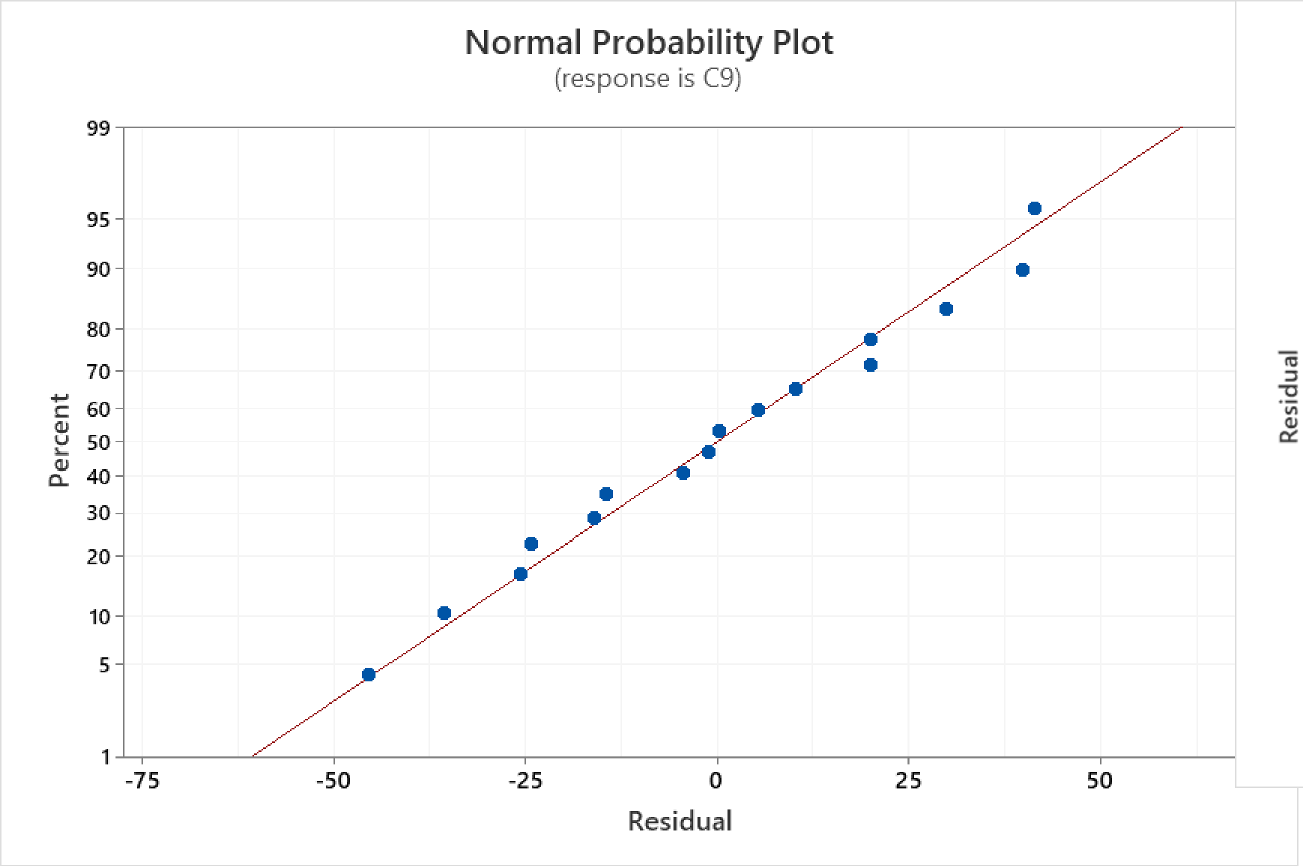
Model Summary

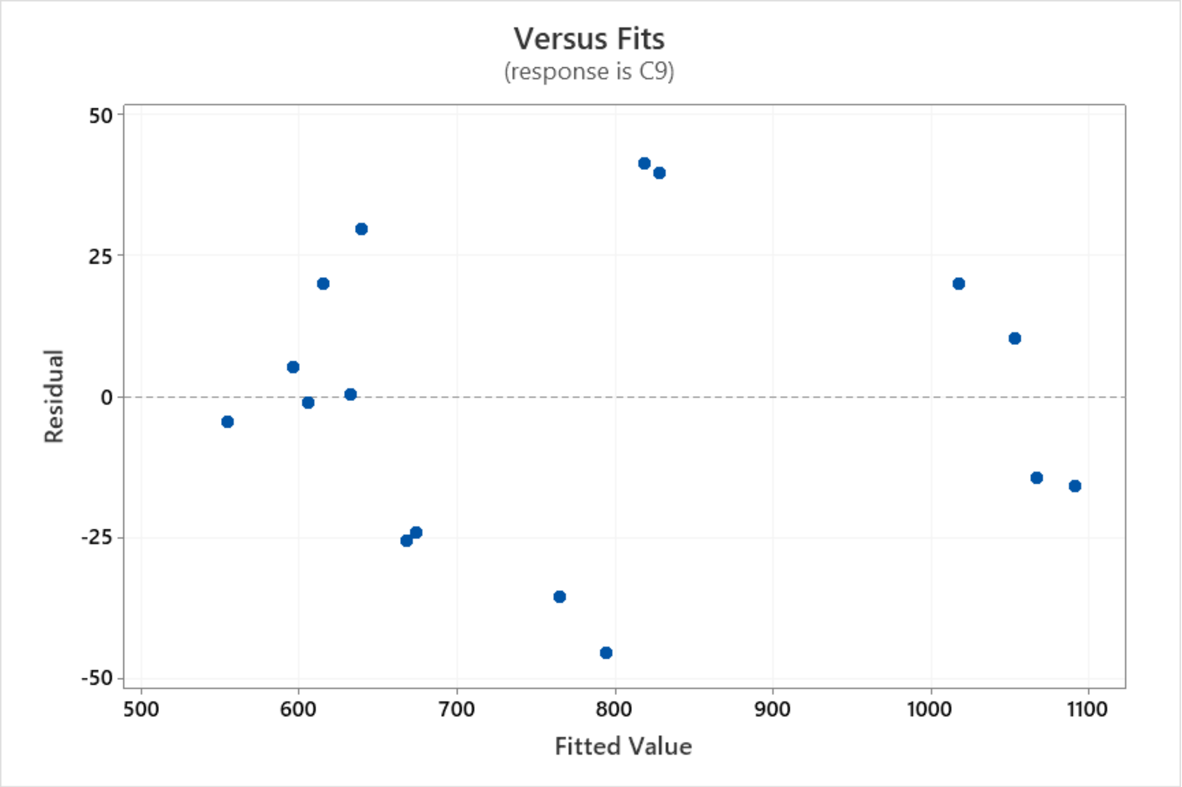
S	R-sq	R-sq(adj)	R-sq(pred)
45.1372	98.08%	94.25%	80.37%

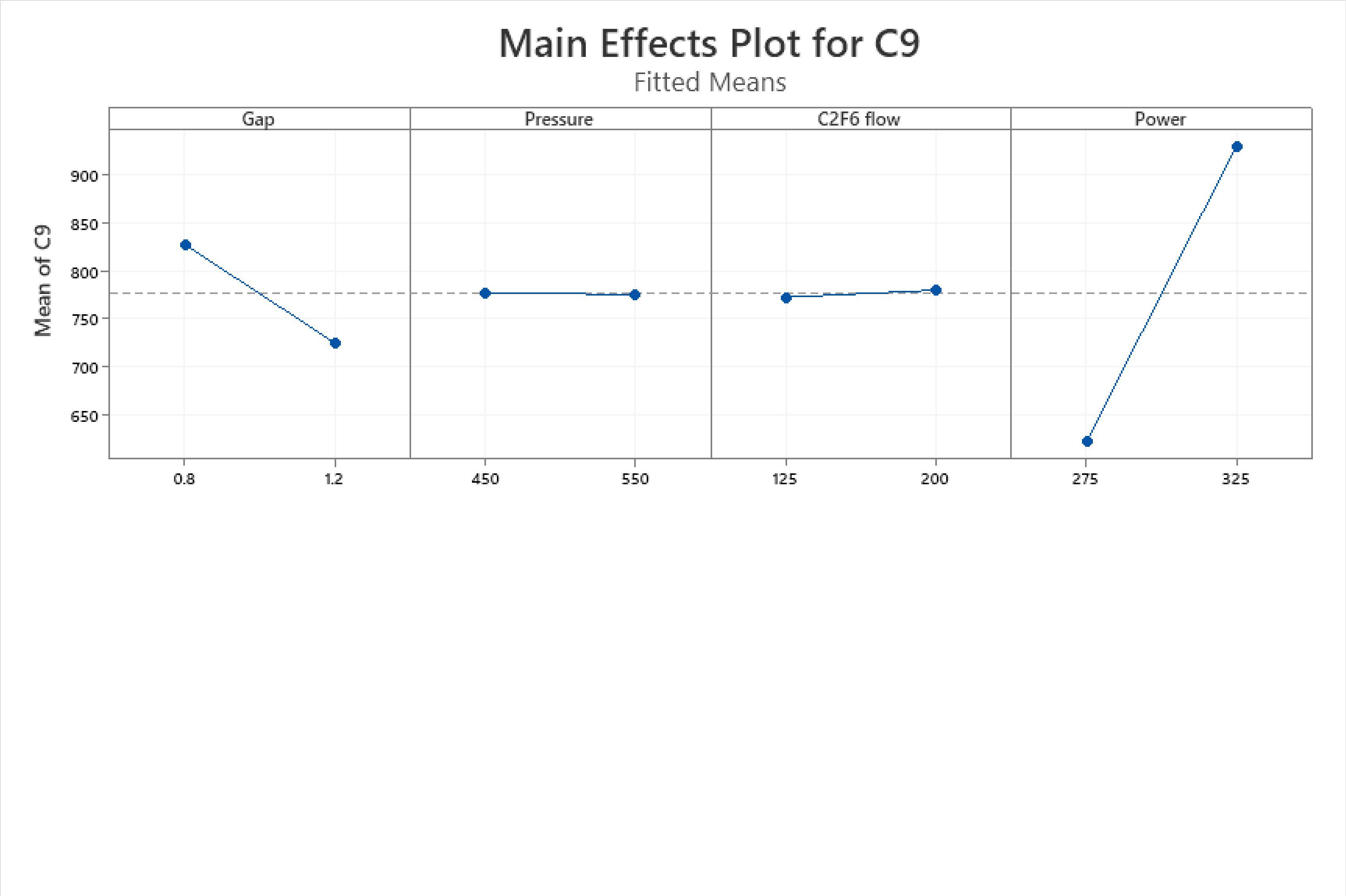
Analysis of Variance

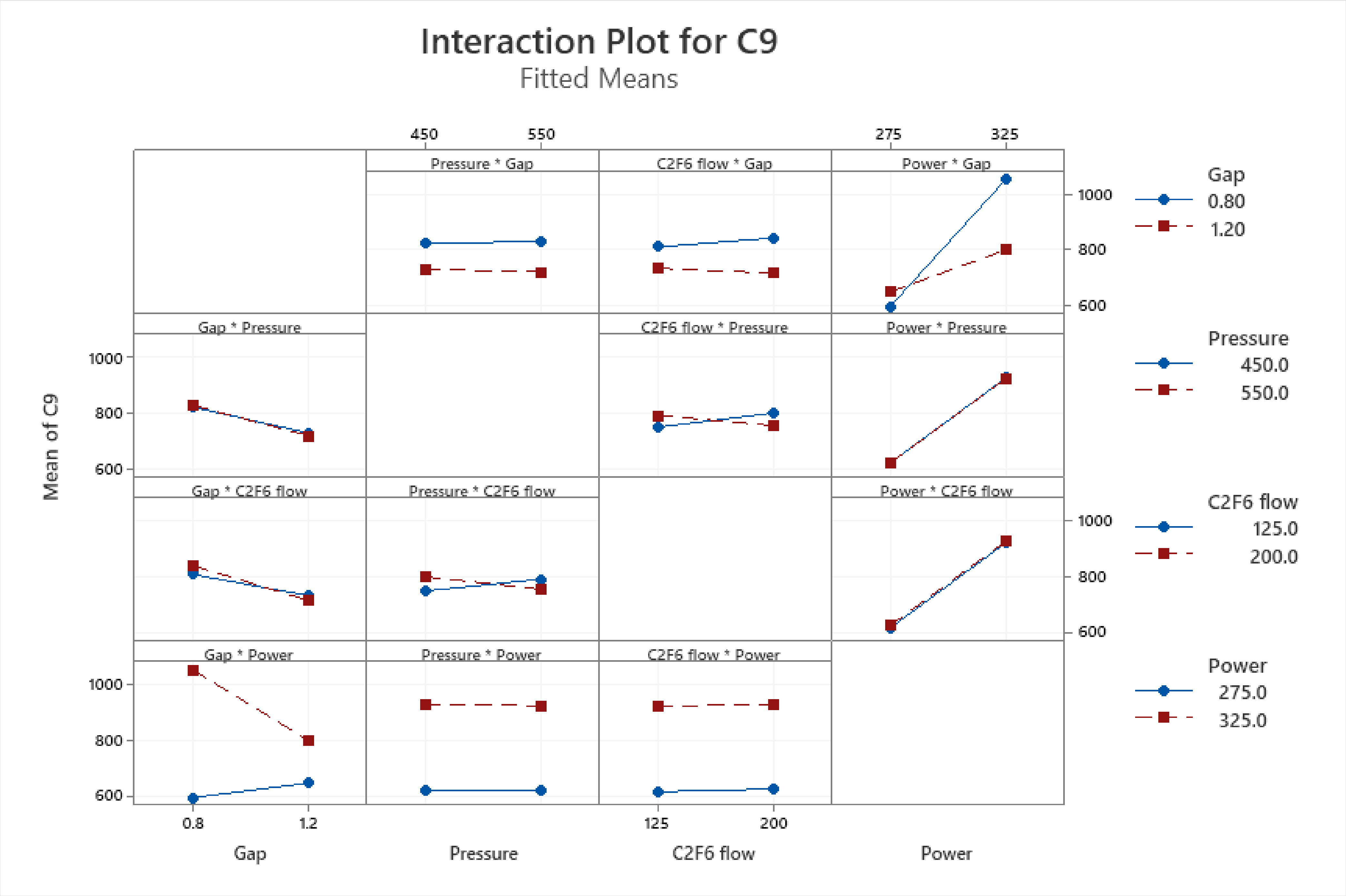
Source	DF	Adj SS	Adj MS	F-Value	P-Value
Model	10	521234	52123	25.58	0.001
Linear	4	416389	104097	51.09	0.000
Gap	1	41311	41311	20.28	0.006
Pressure	1	11	11	0.01	0.945
C2F6 flow	1	218	218	0.11	0.757
Power	1	374850	374850	183.99	0.000
2-Way Interactions	6	104845	17474	8.58	0.016
Gap*Pressure	1	248	248	0.12	0.741
Gap*C2F6 flow	1	2475	2475	1.21	0.321
Gap*Power	1	94403	94403	46.34	0.001
Pressure*C2F6 flow	1	7700	7700	3.78	0.109
Pressure*Power	1	2	2	0.00	0.979
C2F6 flow*Power	1	18	18	0.01	0.929
Error	5	10187	2037		
Total	15	531421			


Pareto Chart of the Standardized Effects
(response is C9, α = 0.05)
Term
2.57
D
AD
A
BC
AC
AB
C
CD
B
BD
0
2
4
6
8
10
12
14
Standardized Effect
Factor Name
A Gap
B Pressure
C C2F6 flow
D Power


Normal Probability Plot
(response is C9)
Percent
99
95
90
80
70
60
50
40
30
20
10
5
1
-75
-50
-25
0
25
50
Residual


Versus Fits
(response is C9)
Residual
50
25
0
-25
-50
500
600
700
800
900
1000
1100
Fitted Value


Main Effects Plot for C9
Fitted Means
Gap
Pressure
C2F6 flow
Power
Mean of C9
900
850
800
750
700
650
0.8
1.2
450
550
125
200
275
325



Interaction Plot for C9
Fitted Means
Pressure * Gap
C2F6 flow * Gap
Power * Gap
Gap * Pressure
C2F6 flow * Pressure
Power * Pressure
Gap * C2F6 flow
Pressure * C2F6 flow
Power * C2F6 flow
Gap * Power
Pressure * Power
C2F6 flow * Power
450
550
275
325
0.8
1.2
125
200
600
800
1000
Mean of C9
Gap
Pressure
C2F6 flow
Power
Gap
0.80
1.20
Pressure
450.0
550.0
C2F6 flow
125.0
200.0
Power
275.0
325.0

Addition of Center Points to a 2^k Design

A potential concern in the use of two-level factorial designs is the assumption of linearity in the factor effects. Of course, perfect linearity is unnecessary, and the 2^k system works quite well even when the linearity assumption holds only approximately. However, a method of replicating certain points in the 2^k factorial provides protection against curvature and allows an independent estimate of error to be obtained. The method consists of adding **center points** to the 2^k design. These consist of n_C replicates run at the point $x_i = 0$ ($i = 1, 2, \ldots, k$). One important reason for adding the replicate runs at the design center is that center points do not affect the usual effects estimates in a 2^k design. We assume that the k factors are quantitative.

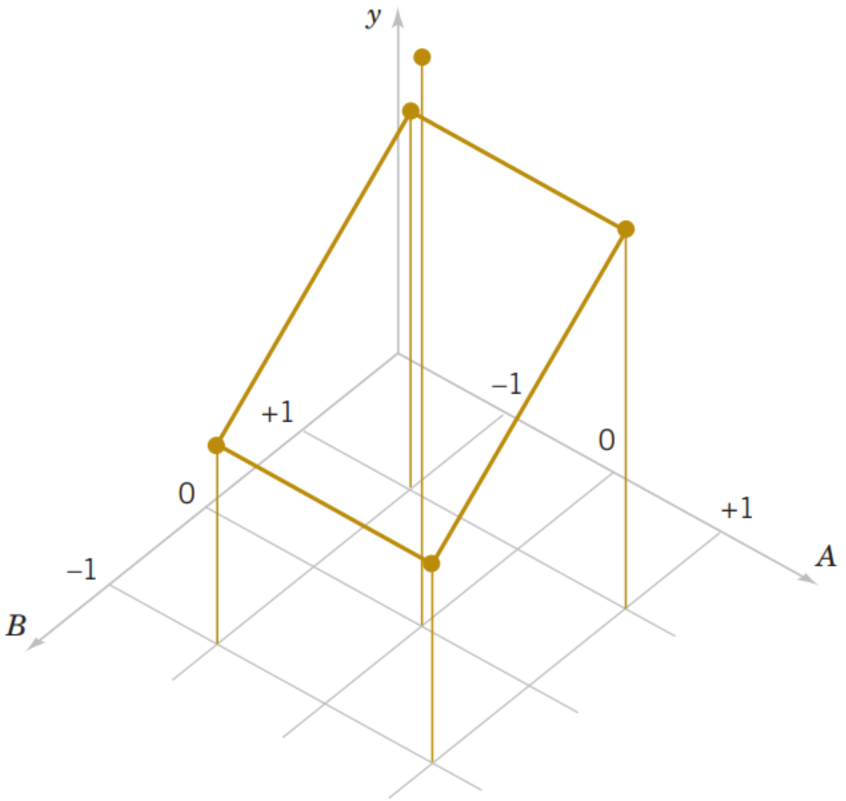


FIGURE 14-26 A 2^2 design with center points.

$$SS_{\text{Curvature}} = \frac{n_F n_C \left(\bar{y}_F - \bar{y}_C\right)^2}{n_F + n_C} = \left(\frac{\bar{y}_F - \bar{y}_C}{\sqrt{\frac{1}{n_F} + \frac{1}{n_C}}} \right)^2$$

$$t\text{-statistic for Curvature} = \frac{\bar{y}_F - \bar{y}_C}{\hat{\sigma}\sqrt{\frac{1}{n_F} + \frac{1}{n_C}}}$$

When points are added to the center of the 2^k design, the model we may entertain is

$$Y = \beta_0 + \sum_{j=1}^{k} \beta_0 x_j + \sum_{i<j}\sum \beta_{ij} x_i x_j + \sum_{j=1}^{k} \beta_{jj} x^2{}_j + \epsilon$$

where the β_{jj} are pure quadratic effects. The test for curvature actually tests the hypotheses

$$H_0\colon \sum_{j=1}^{k} \beta_{jj} = 0 \qquad H_1\colon \sum_{j=1}^{k} \beta_{jj} \neq 0$$

Furthermore, if the factorial points in the design are unreplicated, we may use the n_C center points to construct an estimate of error with $n_C - 1$ degrees of freedom.

Addition of Center Points to a 2^k Design

Example 14-6

Process Yield A chemical engineer is studying the percentage of conversion or yield of a process. There are two variables of interest, reaction time and reaction temperature. Because she is uncertain about the assumption of linearity over the region of exploration, the engineer decides to conduct a 2^2 design (with a single replicate of each factorial run) augmented with five center points. The design and the yield data are shown in Fig. 14-27.

Table 14-22 summarizes the analysis for this experiment. The mean square error is calculated from the center points as follows:

$$MS_E = \frac{SS_E}{n_C - 1} = \frac{\sum_{\text{Center points}} (y_i - \bar{y}_C)^2}{n_C - 1} = \frac{\sum_{i=1}^{5} (y_i - 40.46)^2}{4} = \frac{0.1720}{4} = 0.0430$$

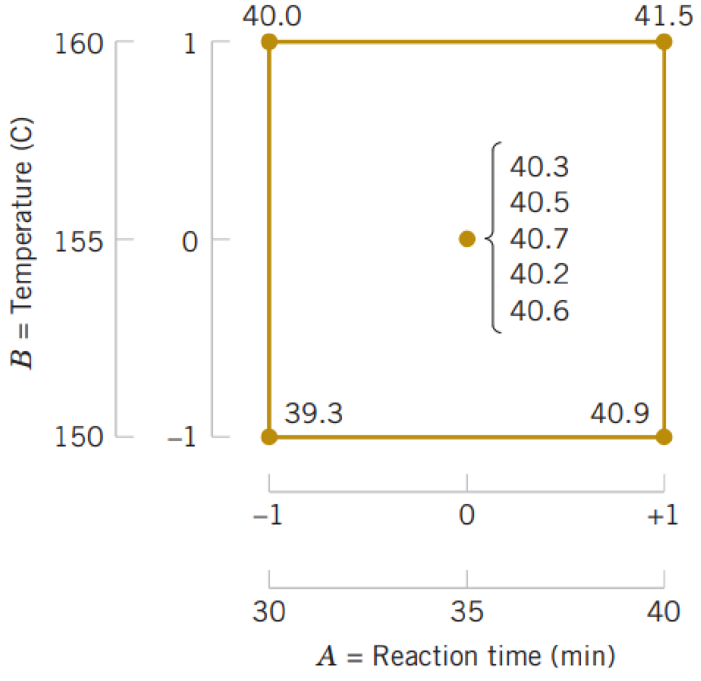


FIGURE 14-27 The 2^2 design with five center points for the process yield experiment in Example 14-6.

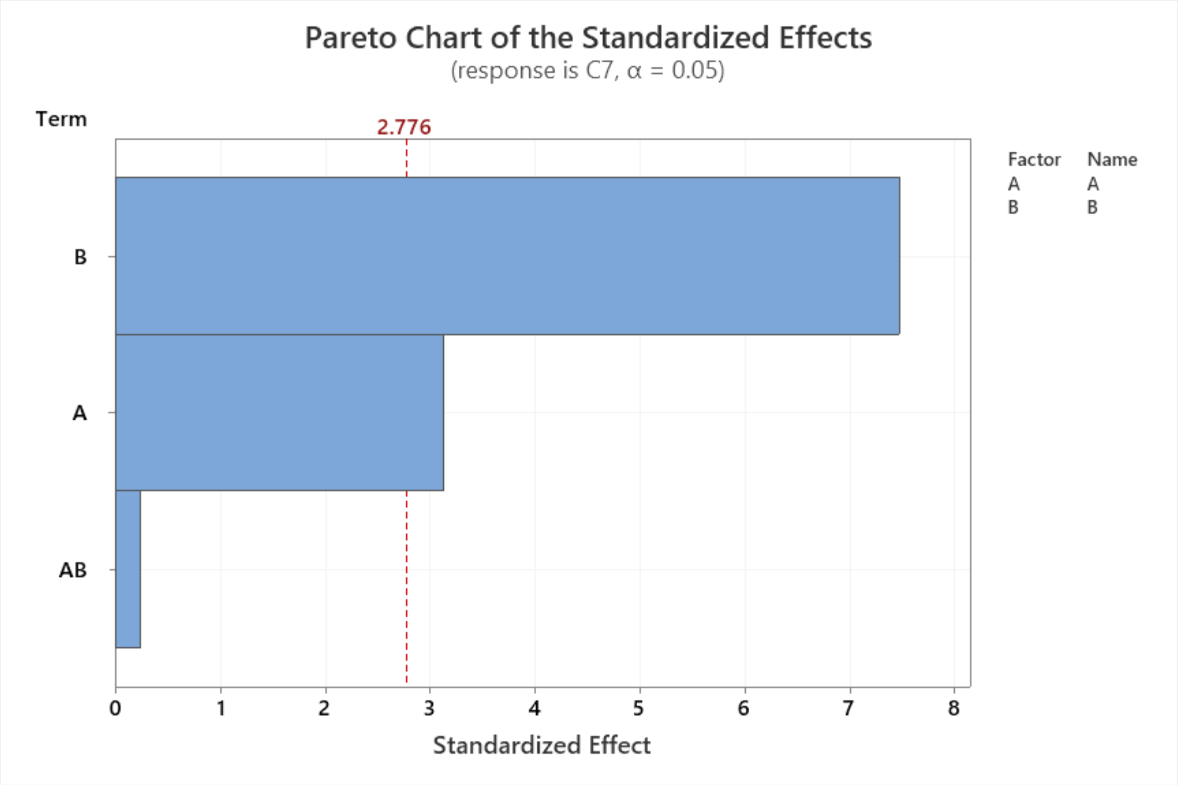
Coded Coefficients

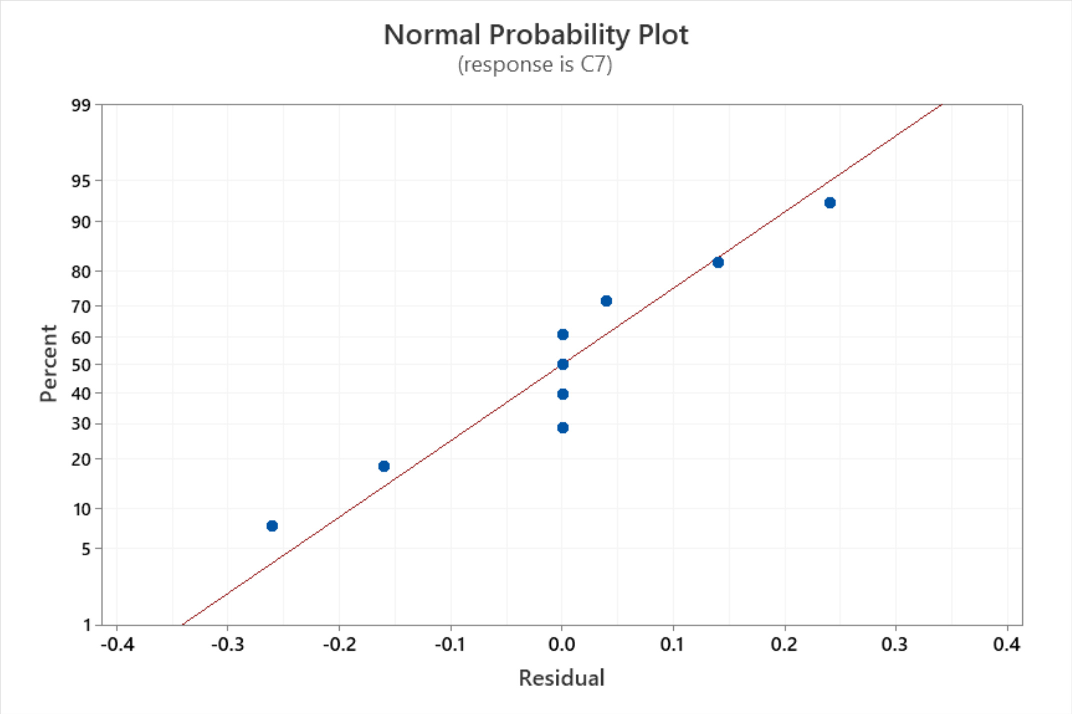
Term	Effect	Coef	SE Coef	T-Value	P-Value	VIF
Constant		40.425	0.104	389.89	0.000	
A	0.650	0.325	0.104	3.13	0.035	1.00
B	1.550	0.775	0.104	7.47	0.002	1.00
A*B	-0.050	-0.025	0.104	-0.24	0.821	1.00
Ct Pt		0.035	0.139	0.25	0.814	1.00

Analysis of Variance

Source	DF	Adj SS	Adj MS	F-Value	P-Value
Model	4	2.83022	0.70756	16.45	0.009
Linear	2	2.82500	1.41250	32.85	0.003
A	1	0.42250	0.42250	9.83	0.035
B	1	2.40250	2.40250	55.87	0.002
2-Way Interactions	1	0.00250	0.00250	0.06	0.821
A*B	1	0.00250	0.00250	0.06	0.821
Curvature	1	0.00272	0.00272	0.06	0.814
Error	4	0.17200	0.04300		
Total	8	3.00222			

Practical Interpretation: The analysis of variance indicates that both factors exhibit significant main effects, that there is no interaction, and that there is no evidence of curvature in the response over the region of exploration. That is, the null hypothesis $H_0 : \sum_{j=1}^{k} \beta_{jj} = 0$ cannot be rejected.


Pareto Chart of the Standardized Effects
(response is C7, α = 0.05)
Term
2.776
B
A
AB
Factor Name
A A
B B
0 1 2 3 4 5 6 7 8
Standardized Effect


Normal Probability Plot
(response is C7)
Percent
99 95 90 80 70 60 50 40 30 20 10 5 1
-0.4 -0.3 -0.2 -0.1 0.0 0.1 0.2 0.3 0.4
Residual

Blocking and Confounding in the 2^k Design

- Ideal: Run all experiments in a 2^k factorial design under homogenous conditions.
- The "ideal" often is not possible. Experiment will often be "blocked" according to some extraneous factor:
    - Multiple equipment setups
    - Different personnel
    - Different raw materials
    - Temporal conditions change

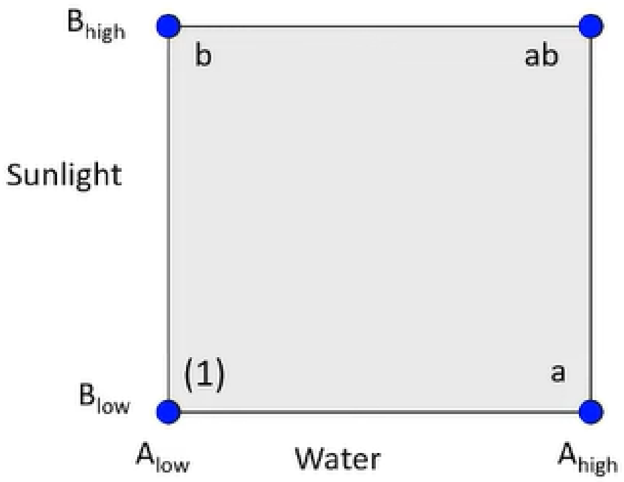
- Block size may be smaller than the number of runs in a complete replicate

- In analyzing the results, the block effect will be confounded with certain factor effects.

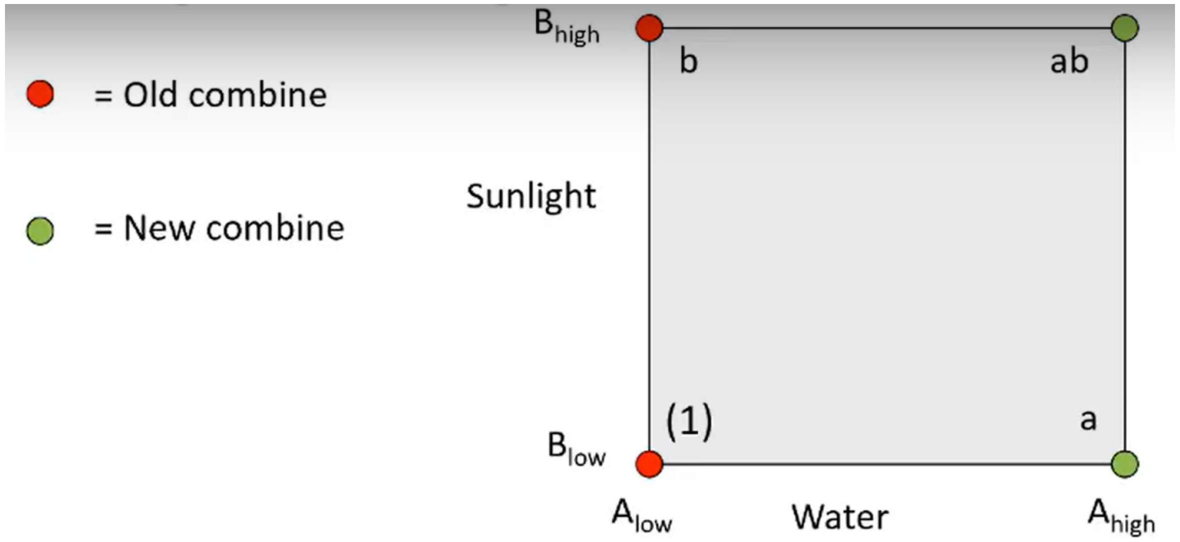
Example: You are agricultural scientist investigating productivity of wheat.

A: Watering (low/high)
B: Sunlight (low/high)
Response: Wheat produced /day

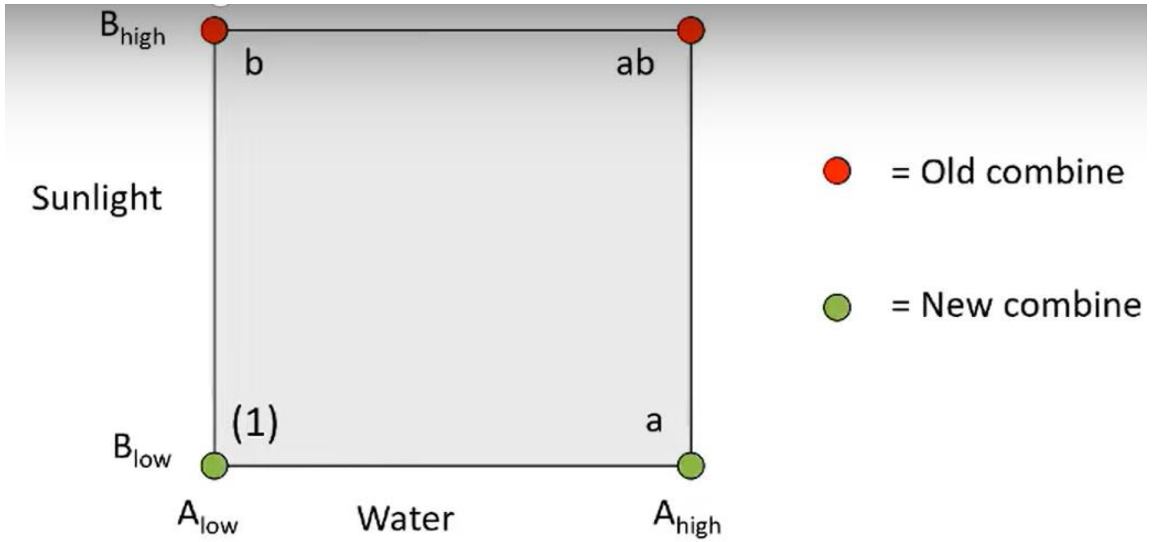
<table>
<tr><td></td><td colspan="2">Watering (A)</td></tr>
<tr><td rowspan="2">Sunlight (B)</td><td>Low watering
Low sunlight
(1) = 50 kg/day</td><td>High watering
Low sunlight
(a) = 150kg/day</td></tr>
<tr><td>Low watering
High sunlight
(b) = 90 kg/day</td><td>High watering
High sunlight
(ab) =170 kg/day</td></tr>
</table>



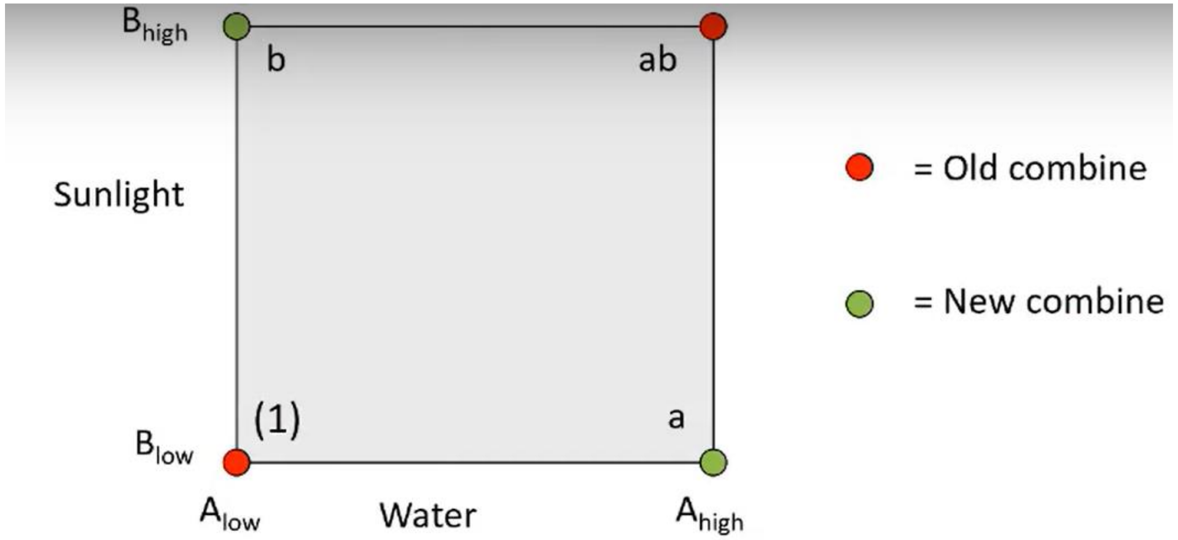
What if there are two different combines that harvest the wheat?



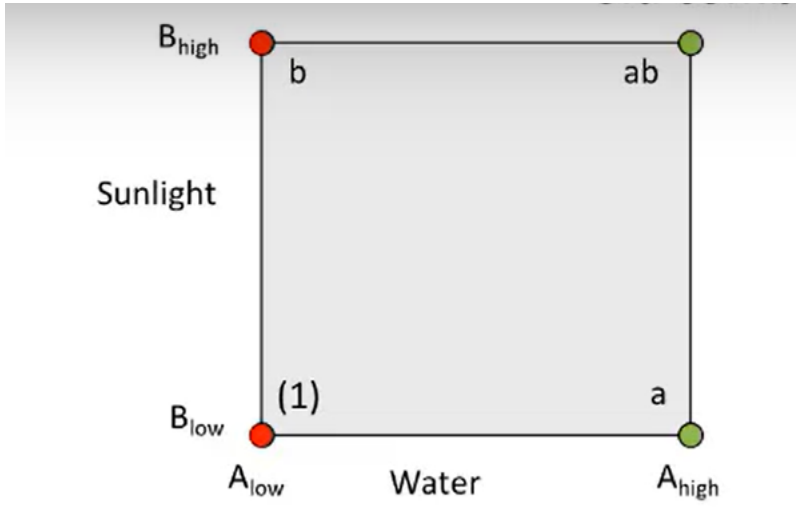
= Old combine
= New combine
Sunlight
B_{high}
b
ab
(1)
a
B_{low}
A_{low}
Water
A_{high}



Sunlight
B_{high}
b
ab
(1)
a
B_{low}
A_{low}
Water
A_{high}
= Old combine
= New combine

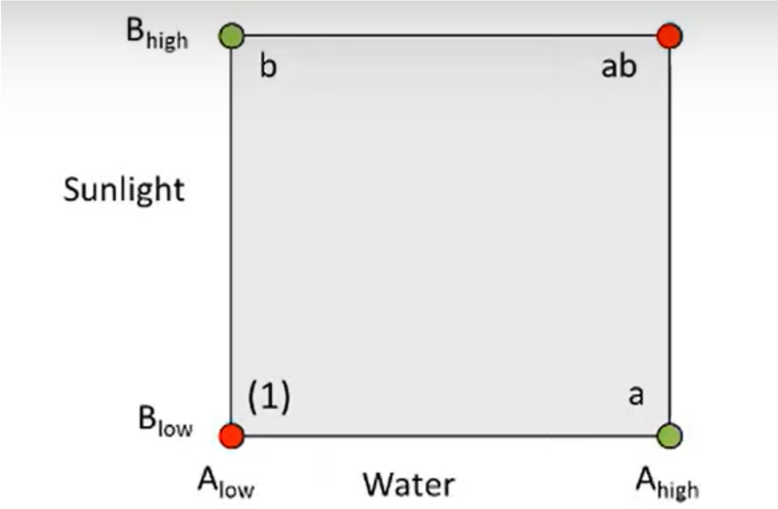


Sunlight
B_{high}
b
ab
(1)
a
B_{low}
A_{low}
Water
A_{high}
= Old combine
= New combine



Contrasts

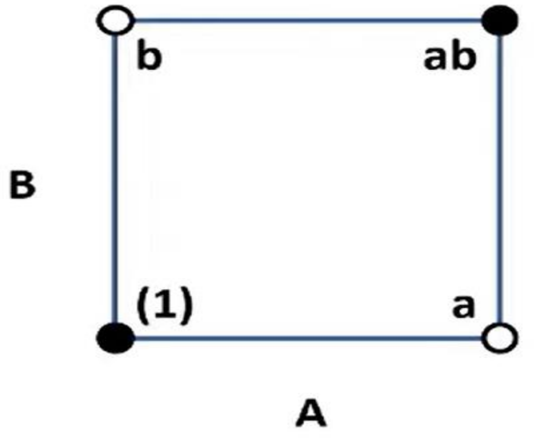
A	$(ab+\delta) + (a+\delta) - b - (1)$
B	$(ab+\delta) + b - (a+\delta) - (1)$
AB	$(ab+\delta) + (1) - (a+\delta) - b$



Contrasts

A	$ab + (a+\delta) - (b+\delta) - (1)$
B	$ab + (b+\delta) - (a+\delta) - (1)$
AB	$ab + (1) - (a+\delta) - (b+\delta)$

δ increase productivity between old and new combine



Runs

● Block 1	○ Block 2
(1)	a
ab	b

Usually confound with the highest order interaction (AB here):

	A	B	AB
(1)	-1	-1	+1
a	+1	-1	-1
b	-1	+1	-1
ab	+1	+1	+1

	A	B	AB
(1)	-1	-1	+1
a	+1	-1	-1
b	-1	+1	-1
ab	+1	+1	+1

$$L = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n$$

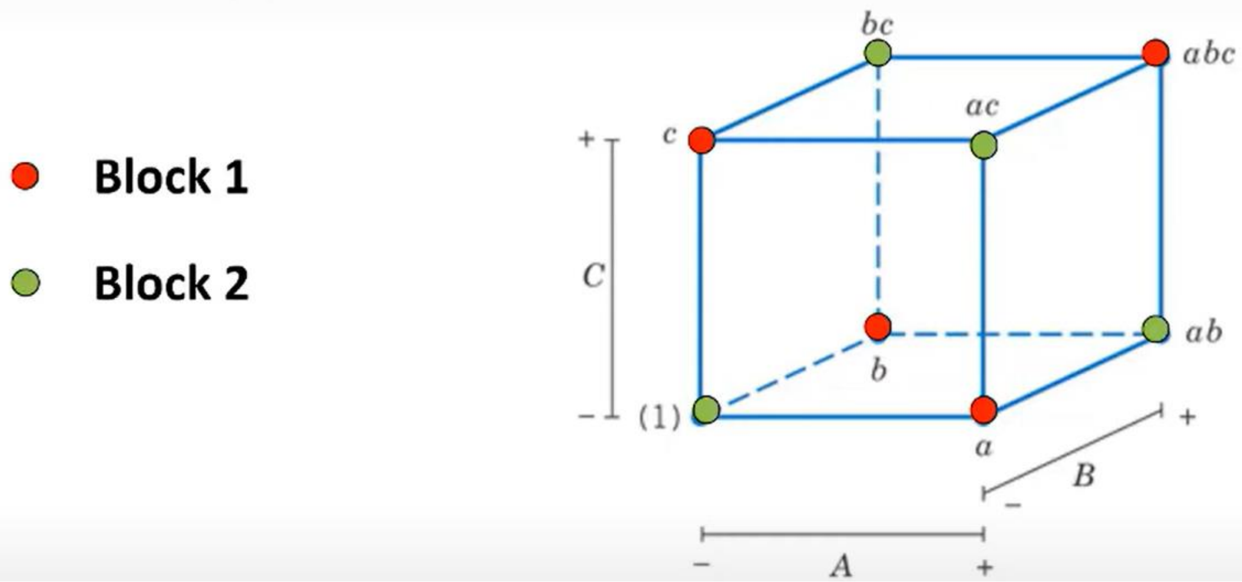
(1): $L = 1(0) + 1(0) = 0 = 0$

a: $L = 1(1) + 1(0) = 1 = 1$

b: $L = 1(0) + 1(1) = 1 = 1$

ab: $L = 1(1) + 1(1) = 2 = 0$

For a 2^3 design, how should we choose which experimental treatments go in each block?



$$L = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n$$

$$(1)\text{: } L = 1(0)+1(0)+1(0) = 0 = 0 \text{ (mod 2)}$$
$$a\text{: } L = 1(1)+1(0)+1(0) = 1 = 1 \text{ (mod 2)}$$
$$b\text{: } L = 1(0)+1(1)+1(0) = 1 = 1 \text{ (mod 2)}$$
$$ab\text{: } L = 1(1)+1(1)+1(0) = 2 = 0 \text{ (mod 2)}$$
$$c\text{: } L = 1(0)+1(0)+1(1) = 1 = 1 \text{ (mod 2)}$$
$$ac\text{: } L = 1(1)+1(0)+1(1) = 2 = 0 \text{ (mod 2)}$$
$$bc\text{: } L = 1(0)+1(1)+1(1) = 2 = 0 \text{ (mod 2)}$$
$$abc\text{: } L = 1(1)+1(1)+1(1) = 3 = 1 \text{ (mod 2)}$$

Treatment Combination	A	B	C	AB	AC	BC	ABC
a	1	-1	-1	-1	-1	1	1
b	-1	1	-1	-1	1	-1	1
c	-1	-1	1	1	-1	-1	1
abc	1	1	1	1	1	1	1
(1)	-1	-1	-1	1	1	1	-1
ab	1	1	-1	1	-1	-1	-1
ac	1	-1	1	-1	1	-1	-1
bc	-1	1	1	-1	-1	1	-1

A shortcut method is useful in constructing these designs. The block containing the treatment combination (1) is called the **principal block**. Any element [except (1)] in the principal block may be generated by multiplying two other elements in the principal block modulus 2 on the exponents. For example, consider the principal block of the 2^3 design with ABC confounded, shown in Fig. 14-29. Note that

$$ab \cdot ac = a^2bc = bc$$

$$ab \cdot bc = ab^2c = ac$$

$$ac \cdot bc = abc^2 = ab$$

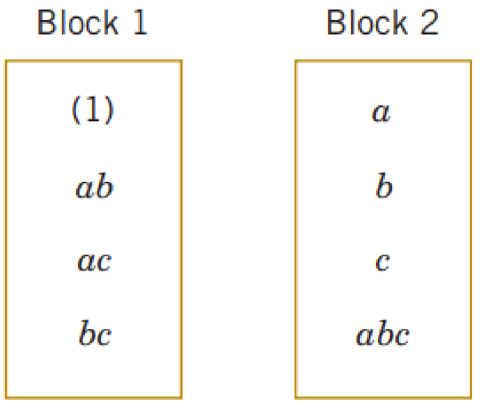
Treatment combinations in the other block (or blocks) may be generated by multiplying one element in the new block by each element in the principal block modulus 2 on the exponents. For the 2^3 with ABC confounded, because the principal block is (1), ab, ac, and bc, we know that the treatment combination b is in the other block. Thus, elements of this second block are

$$b \cdot (1) = b$$

$$b \cdot ab = ab^2 = a$$

$$b \cdot ac = abc$$

$$b \cdot bc = b^2c = c$$



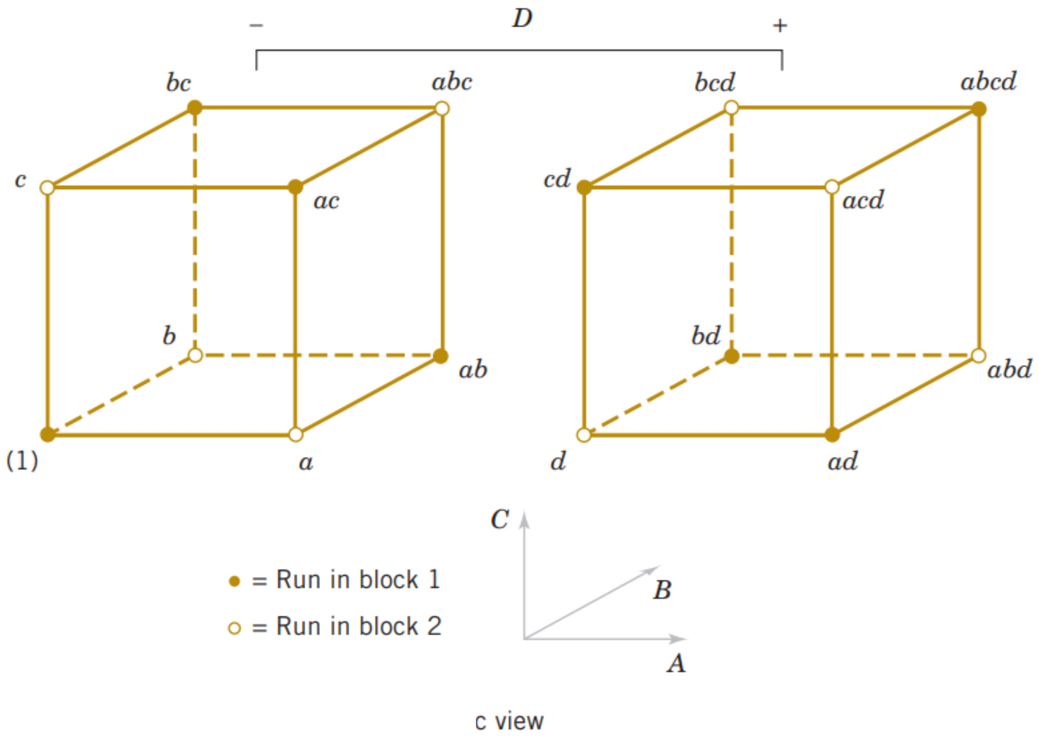
Assignment of the eight runs to two blocks

Example: Researchers reported on an experiment to minimize variation in blood glucose levels. The factors were volume of juice intake before exercise (4-8 g), amount of exercise on a Nordic Track cross-country skier (10-20 min), and delay between time of juice intake (0-20 min) and the beginning of the exercise period.

If you wish to block for time of day (am vs. pm), how should we choose with experiments to conduct in the am vs. pm?

Run	Juice (oz) A	Exercise (min) B	Delay (min) C	Time of Day
1	4	10	0	
2	8	10	0	
3	4	20	0	
4	8	20	0	
5	4	10	20	
6	8	10	20	
7	4	20	20	
8	8	20	20	

Treatment Combination	A	B	C	AB	AC	BC	ABC
a	1	-1	-1	-1	-1	1	1
b	-1	1	-1	-1	1	-1	1
c	-1	-1	1	1	-1	-1	1
abc	1	1	1	1	1	1	1
(1)	-1	-1	-1	1	1	1	-1
ab	1	1	-1	1	-1	-1	-1
ac	1	-1	1	-1	1	-1	-1
bc	-1	1	1	-1	-1	1	-1



Block 1		Block 2	
(1)	= 3	*a*	= 7
ab	= 7	*b*	= 5
ac	= 6	*c*	= 6
bc	= 8	*d*	= 4
ad	= 10	*abc*	= 6
bd	= 4	*bcd*	= 7
cd	= 8	*acd*	= 9
abcd	= 9	*abd*	= 12

Assignment of the sixteen runs to two blocks

(*b*)

Analysis of Variance

Source	DF	Adj SS	Adj MS	F-Value	P-Value
Model	11	80.6875	7.3352	6.90	0.038
Blocks	1	0.0625	0.0625	0.06	0.820
Linear	4	46.2500	11.5625	10.88	0.020
A	1	27.5625	27.5625	25.94	0.007
B	1	1.5625	1.5625	1.47	0.292
C	1	3.0625	3.0625	2.88	0.165
D	1	14.0625	14.0625	13.24	0.022
2-Way Interactions	6	34.3750	5.7292	5.39	0.062
A*B	1	0.0625	0.0625	0.06	0.820
A*C	1	22.5625	22.5625	21.24	0.010
A*D	1	10.5625	10.5625	9.94	0.034
B*C	1	0.5625	0.5625	0.53	0.507
B*D	1	0.5625	0.5625	0.53	0.507
C*D	1	0.0625	0.0625	0.06	0.820
Error	4	4.2500	1.0625		
Total	15	84.9375			


Lean Six Sigma
DMAIC
D
Define
Define the problem.
M
Measure
Quantify the problem.
A
Analyze
Identify the cause of the problem.
I
Improve
Implement and verify the solution.
C
Control
Maintain the solution.